Lecture 9: Fiber, pulse propagation, dispersion.

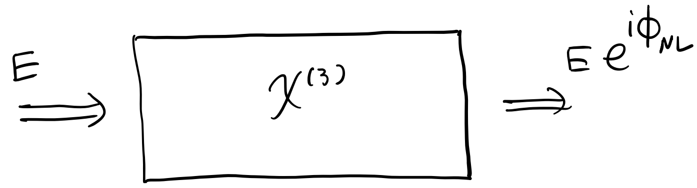
Learning objectives:

1. pulses, wave packets
2. Pulse propagation eq. in nonlinear media
3. Different regimes for pulse propagation

$$\begin{cases} \text{Dispersion length } L_D \\ \text{Nonlinear length } L_{NL} \end{cases}$$

4. Group velocity dispersion (GVD)

3^{rd} order nonlinearity:

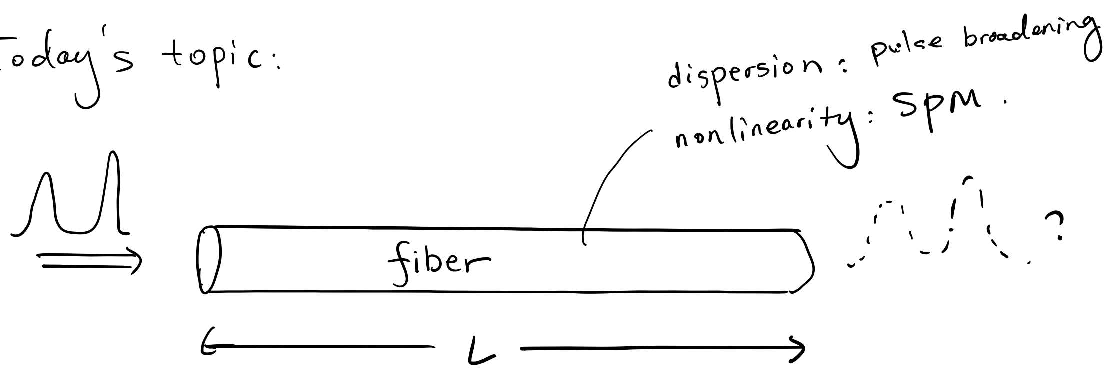


$$n = n_0 + n_2 I$$

$$\Downarrow$$

$$n(t) = n_0 + n_2 I(t)$$

Today's topic:

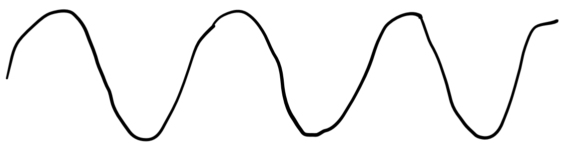


Goal: Derive a pulse propagation equation in fibers or waveguides.

1. Pulse / wave packet

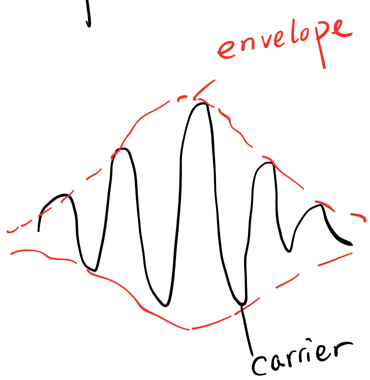
Plane wave:

single frequency.



$$\psi(z,t) = A e^{i(kz-\omega t)}$$

Wavepacket



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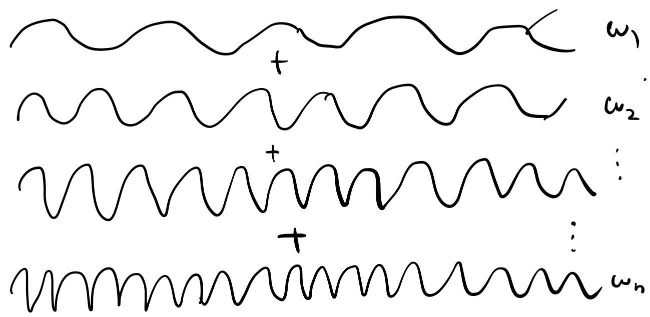
Superposition of plane waves



$$\psi(z,t) = \sum_n A_n e^{i(k_n z - \omega_n t)}$$

2. Pulse propagation equation in nonlinear media

$$\nabla^2 E(z,t) - \mu_0 \epsilon(\omega) \frac{\partial^2 E(z,t)}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

$$\Rightarrow \nabla^2 E(z,t) - \mu_0 \epsilon(\omega) \frac{\partial^2 E(z,t)}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

where $E(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(z,\omega) e^{-i\omega t} d\omega$.

$$\Rightarrow \frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + \underbrace{\mu_0 \epsilon(\omega)\, \omega^2}_{k(\omega)} \tilde{E}(z,\omega) = -\mu_0 \omega_0^2 P_{NL}(z,\omega)$$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + k^2(\omega) \tilde{E}(z,\omega) = -\mu_0 \omega_0^2 \tilde{P}_{NL}(z,\omega)}$$

For pulses, we need to study how the envelope evolves!

Now, let's encode the envelop stuff on the E-field.

$$E(z,t) = \underbrace{A(z,t)}_{\text{envelop}} \cdot e^{i(k_0 z - \omega_0 t)}$$

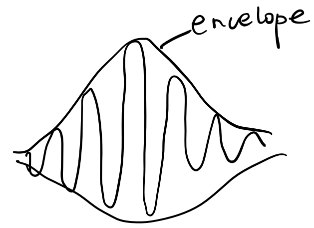
normal phase of plane wave

$$\underbrace{A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z, \delta\omega) e^{-i\delta\omega t} d\delta\omega}_{\text{Sum of all freq. components around the central freq.}} \qquad \delta\omega = \omega - \omega_0$$

central freq.

Then,

$$E(z,t) = \frac{e^{i(k_0 z - \omega_0 t)}}{2\pi} \int_{-\infty}^{\infty} \hat{A}(z,\delta\omega) e^{-i\delta\omega t} \, d(\delta\omega) \quad ①$$



Also, we know

$$E(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(z,\omega) e^{-i\omega t} \, d\omega. \quad ②$$

Compare ① and ②, we get.

$$\tilde{E}(z,\omega) = \hat{A}(z,\omega-\omega_0) e^{ik_0 z}.$$

Recall:

$$\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + k^2(\omega) \tilde{E}(z,\omega) = -\mu_0 \omega_0^2 \tilde{P}_{NL}$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} \tilde{A} + 2ik_0 \frac{\partial \tilde{A}}{\partial z} + (k(\omega)^2 - k_0^2) \tilde{A} = -\mu_0 \omega_0^2 \tilde{P}_{NL}$$

using slowly vary envelope approx. (SVEA). $\frac{\partial^2}{\partial z^2} A << \frac{\partial A}{\partial z}$.

$$\Rightarrow 2ik_0 \frac{\partial \tilde{A}}{\partial z} + [k^2(\omega) - k_0^2] \tilde{A} = -\mu_0 \omega_0^2 \tilde{P}_{NL}$$

$$(k(\omega) + k_0)(k(\omega) - k_0) \simeq 2k_0 [k(\omega) - k_0]$$

$$\Rightarrow i \frac{\partial \tilde{A}}{\partial z} + [k(\omega) - k_0] \tilde{A} = -\frac{\mu_0 \omega_0^2 \tilde{P}_{NL}}{2k_0}$$

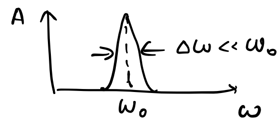
let's expand it!

$$i\frac{\partial \tilde{A}}{\partial z} + \Big[k_0 + \underbrace{\frac{\partial k}{\partial \omega}\Big|_{\omega_0}(\omega-\omega_0)}_{1^{st}\text{ order}} + \underbrace{\frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\Big|_{\omega_0}(\omega-\omega_0)^2}_{2^{nd}\text{ order}} \underbrace{+ \cdots}_{\text{higher order}} - k_0\Big]\tilde{A} = -\frac{\mu_0\omega_0^2 \tilde{P}_{NL}}{2k_0} \quad (3)$$

Comments:

1. CW light, single freqency ω_0. 2^{nd} order can be ignored.

2. pulses (> 1ps)



   $k(\omega)$ can be approximated by 1^{st} order and 2^{nd} order.

3. For even shorter pulses (< 100fs), higher order terms matter!

Operators:

$$(\omega-\omega_0) \equiv i\frac{\partial}{\partial t}\ ; \qquad (\omega-\omega_0)^2 = i^2\frac{\partial^2}{\partial t^2}$$

Also,
$$\frac{\partial k}{\partial \omega} = \frac{1}{v_g}\ , \qquad \frac{\partial^2 k}{\partial \omega^2} = \beta_2$$

Then eq. (3) can be written as

$$\frac{\partial \tilde{A}}{\partial z} + \frac{1}{v_g}\cdot\frac{\partial \tilde{A}}{\partial t} + \frac{i}{2}\beta_2\frac{\partial^2 \tilde{A}}{\partial t^2} = \frac{i\mu_0\omega_0^2}{2k_0}\tilde{P}_{NL}$$

Tranform $\tilde{A}(\omega, z)$ to $A(\omega, t)$.

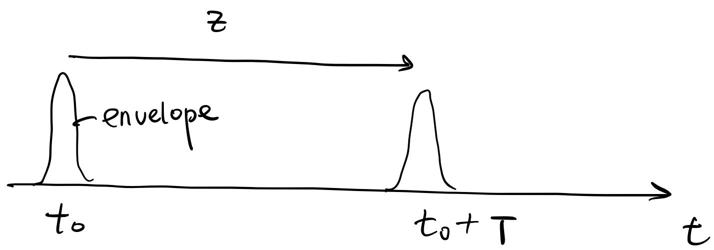
i.e. $A(t) = \int_{-\infty}^{\infty} \tilde{A}(\omega) e^{-i\omega t} d\omega$

$$\frac{\partial A}{\partial z} + \frac{1}{v_g}\frac{\partial A}{\partial t} + \frac{i}{2}\beta_2 \frac{\partial^2 A}{\partial t^2} = i\,\frac{\mu_0 \omega_0^2}{2k_0} P_{NL}$$

For Kerr effect, $P_{NL} = 3\epsilon_0 \chi^{(3)} |A|^2 \cdot A$

$$\Rightarrow \frac{\partial A}{\partial z} + \frac{1}{v_g}\frac{\partial A}{\partial t} + \frac{i}{2}\beta_2 \frac{\partial^2 A}{\partial t^2} = i\gamma |A|^2 \cdot A.$$

$$\gamma = \frac{3\omega_0^2}{2k_0 c^2}\chi^{(3)} = 2\bar{n}_2 \frac{\omega_0}{c}$$



Define $T = t - \frac{z}{v_g}$, then we get:

$$\boxed{\frac{\partial A}{\partial z} + \frac{i}{2}\beta_2 \frac{\partial^2 A}{\partial T^2} = i\gamma |A|^2 \cdot A}$$

Pulse propagation eq in nonlinear media (4)

Dispersion

nonlinearity (Kerr)

Comments:

1. Eq. (4) generalizes all transverse (spatial) modes in fiber / waveguides.

   k_0, → wave numer of interest ; $k(\omega) \Rightarrow$ modal dispersion

   γ → account for the effective mode area

2. In bulk media

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d\left(n(\omega)\frac{\omega}{c}\right)}{d\omega} = \frac{1}{c}\underbrace{\left[n + \omega\frac{dn}{d\omega}\right]}_{n_g}$$

$$\beta_2 = \frac{d^2k}{d\omega^2} = \frac{1}{c}\left[2\frac{dn}{d\omega} + \omega\frac{d^2n}{d\omega^2}\right] = \frac{1}{c}\frac{dn_g}{d\omega} = \frac{1}{d\omega}\left(\frac{1}{v_g}\right)$$

= group velocity dispersion (GVD) parameter

3. Easy to incorporate loss:

$$\frac{\partial A}{\partial z} + \alpha A + \frac{1}{2}i\beta_2\frac{\partial^2 A}{\partial t^2} = i\gamma|A|^2 \cdot A$$

losses

3. Different regimes for pulse propagation

Next question: dispersion dominant?

Nonlinearity dominant?

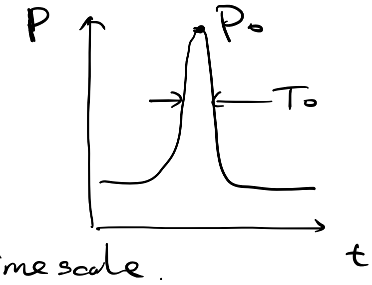
Chapter 2-3

Agarwal.

Define: T_0 = pulse width.

P_0 = pulse peak power

$\tau = \frac{T}{T_0} = \frac{t - z/v_g}{T_0}$ Normalized timescale.



Pulse envelope:

propagation loss

$$A(z, \tau) = \sqrt{P_0}\exp\left(-\frac{\alpha z}{2}\right)\cdot U(z, \tau),$$ plug in ④.

nomalized amplitude.

$$\rightarrow \quad i\frac{\partial U}{\partial z} = \frac{\mathrm{sgn}(\beta_2)}{2L_D}\frac{\partial^2 U}{\partial \tau^2} - \frac{\exp(-\alpha z)}{L_{NL}}|U|^2 U$$

where $L_D = \frac{T_0^2}{|\beta_2|}$ = dispersion length

$L_{NL} = \frac{1}{\gamma P_0}$ = nonlinear length

Regime 1: $L \ll L_D$; $L \ll L_{NL}$

Both dispersion and nonlinearity are not playing a significant role. pulse maintains its shape during propagation

example: telecom, $L \sim 50\,km$, L_D and L_{NL} should $> 500\,km$.

at $1.55\,\mu m$, for silica fibers, $|\beta_2| \simeq 20\,ps^2/km$. $\gamma \simeq 2W^{-1}km^{-1}$

If $P_0 < 1mW$, $T_0 > 100ps$. ok!

Regime 2: $L \ll L_{NL}$; $L \simeq L_D$. , $\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \ll 1$

Dispersion is dominant, pulse gets broadened

example: $T_0 = 1\,ps$, $P_0 \ll 1\,W$.

Regime 3: $L \ll L_D$, $L \simeq L_{NL}$. $\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \gg 1$

Kerr nonlinearity is dominant $\Rightarrow$ self-phase modulation, spectral broadening

example: $T_0 = 100\,ps$, $P_0 > 1\,W$.

Regime 4: $L \gg L_D$; $L \gg L_{NL}$.

Both nonlinearity and dispersion are playing important roles.

If $\beta_2 < 0$. (anomalous dispersion) $\Rightarrow$ soliton

$\beta_2 > 0$. pulse compression

4. Group velocity dispersion

Set $\gamma = 0$, consider dispersion only.

$$\frac{\partial A}{\partial z} = -\frac{1}{2} i \beta_2 \frac{\partial^2 A}{\partial T^2}$$

$$A(z,T) = \frac{1}{2\pi}\int \tilde{A}(z,\omega)\, e^{-i\omega T}\, d\omega.$$

$$\frac{\partial \tilde{A}}{\partial z} = \frac{i}{2}\beta_2 \omega^2 \tilde{A}. \Rightarrow \tilde{A}(z,\omega) = \tilde{A}(0,\omega)\, e^{\frac{i}{2}\beta_2\omega^2 z}$$

$$\Rightarrow A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(0,\omega)\, e^{i(k_2\omega^2 z/2 - \omega T)}\, d\omega$$

$$\tilde{A}(0,\omega) = \int_{-\infty}^{\infty} A(0,T)\, e^{i\omega T}\, dT$$

Example. Gaussian pulse.

$$A(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right), \quad T_{FWHM} = 2\sqrt{\ln 2}\, T_0 \simeq 1.665 T_0$$

$$\Rightarrow \tilde{A}(0,\omega) = \int_{-\infty}^{\infty} \exp\left(-\frac{T^2}{2T_0^2}\right) dT.$$

$$\Rightarrow A(z,t) = \frac{T_0}{(T_0^2 - i\beta_2 z)^{1/2}} \exp\left[\underbrace{-\frac{T^2}{2(T_0^2 - i\beta_2 z)}}_{}\right]$$

$$\cdot \left(\frac{T_0^2 + i\beta_2 z}{T_0^2 + i\beta_2 z}\right)$$

$$\Rightarrow A(z,t) = \frac{T_0}{\sqrt{T_0^2 - i\beta_2 z}} e^{-\frac{T^2(T_0^2 + i\beta_2 z)}{2(T_0^2 + \beta_2 z)^2}}$$

$$= \frac{T_0}{\sqrt{T_0^2 - i\beta_2 z}} \cdot \exp \cdot \left[- \frac{T^2 \left(1 + \frac{i\beta_2 z}{T_0^2}\right)}{2\left(T_0^2 + \frac{\beta_2 z}{T_0}\right)^2} \right]$$

Pulse width at z:

$$T_1 = T_0 \sqrt{1 + \left(\frac{\beta_2 z}{T_0^2}\right)^2} = \underbrace{T_0 \sqrt{1 + \left(\frac{z}{L_D}\right)^2}}_{\text{when } z \sim L_D \text{, broadening!}}$$

At $z = L_D$, $T_1 = \sqrt{2}\, T_0$