Lecture 9. Periodic media

Learning objectives:

① periodic media and Bloch theorem.

② Bloch wave and photonic bandgap.

③ Example 1: Bragg reflectors

④ Example 2: transmission filters

1. Periodic media and Bloch theorem (P539. Yariv)

Definition: optical structures whose ϵ or μ are periodic functions of position:

i.e. $\epsilon(x) = \epsilon(x+a)$

$\mu(x) = \mu(x+a)$

The medium "looks" exactly the same at x. and $x+a$

Bloch theorem:

For periodic medium, the solution of Maxwell eqs.

$$\nabla \times \vec{H} = i\omega\epsilon\vec{E}$$
$$\nabla \times \vec{E} = -i\omega\mu\vec{H}$$

is in the form of

Bloch wavevector

$$\left.\begin{aligned} \vec{E} &= \vec{E}_K(\vec{x})e^{-i\vec{k}\cdot\vec{x}} \\ \vec{H} &= \vec{H}_K(\vec{x})e^{-i\vec{k}\cdot\vec{x}} \end{aligned}\right\} \text{Bloch waves}$$

Depend on K

where $\vec{E}_K(\vec{x})$ and $\vec{H}_K(\vec{x})$ are periodic function of x, i.e.

$$\vec{E}_K(\vec{x}) = \vec{E}_K(\vec{x}+\vec{a})$$
$$\vec{H}_K(\vec{x}) = \vec{H}_K(\vec{x}+\vec{a})$$

Mathematic proof: P544. Yariv

Comments:

① Similar to plane waves, propagating waves in waveguides Bloch waves have dispersion characteristics.

$$\omega = \omega(\vec{k})$$

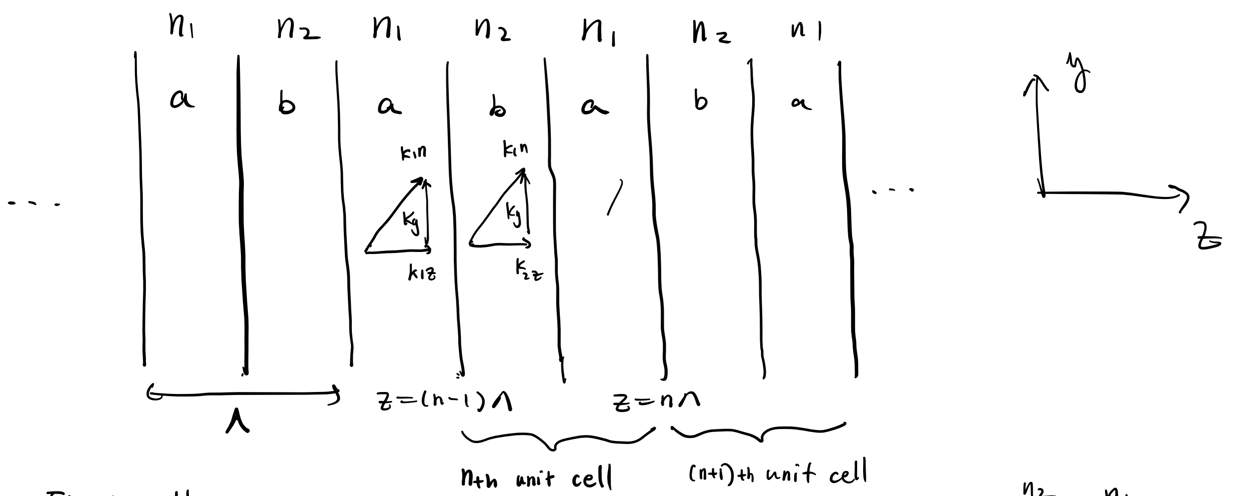
② $\vec{k}$ can be real or complex.

real: propagating Bloch wave

complex: evanescent (decaying) wave.

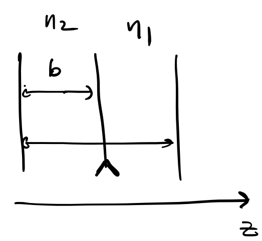
③ By invoking the concept of Bloch wave, the periodic medium can be regarded as a homogeneous waveguide, with wavevector $\vec{k}$

2. Periodic layered media and photonic bandgap



First cell:

$$n(z) = \begin{cases} n_2 & 0 < z < b \\ n_1 & b < z < \Lambda, \end{cases} \quad \text{with} \quad n(z) = n(z+\Lambda)$$



$$\text{Solution: } \vec{E} = \underbrace{\vec{E}(z)}_{\text{medium is not homogeneous along } z} e^{i(\omega t - k_y y)}$$

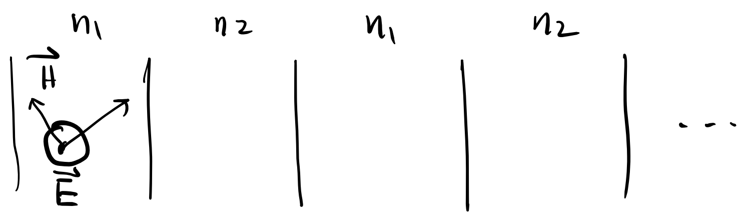
In each homogeneous layer.

$$E(z) = \begin{cases} a_n e^{-ik_{1z}(z-n\Lambda)} + b_n e^{ik_{1z}(z-n\Lambda)}, & n\Lambda - a < z < n\Lambda \\ \underbrace{c_n e^{-ik_{2z}(z-n\Lambda+a)}}_{\text{left prop.}} + \underbrace{d_n e^{+ik_{2z}(z-n\Lambda+a)}}_{\text{right prop.}} & (n-1)\Lambda < z < n\Lambda - a \end{cases}$$

with $k_{1z} = \sqrt{\left(\frac{n_1 \omega}{c}\right)^2 - k_y^2}$

$k_{2z} = \sqrt{\left(\frac{n_2 \omega}{c}\right)^2 - k_y^2}$

① **TE-wave**: (s-wave, $\vec{E}$ vector polarized perpendicular to the y-z plane)



B.C. E_x and H_y are continous at interfaces $z = (n-1)\Lambda$ and $z = n\Lambda - a$.

$$\Rightarrow \begin{cases} a_{n-1} + b_{n-1} = c_n e^{ik_{2z}b} + d_n e^{-ik_{2z}b} \\ ik_{1z}(a_{n-1} - b_{n-1}) = ik_{2z}(c_n e^{ik_{2z}b} - d_n e^{-ik_{2z}b}) \\ c_n + d_n = a_n e^{ik_{1z}a} + b_n e^{-ik_{1z}a} \\ ik_{2z}(c_n - d_n) = ik_{1z}(a_n e^{ik_{1z}a} - b_n e^{-ik_{1z}a}) \end{cases}$$

eliminate c_n, d_n in matrix form $\Longrightarrow$

$$\begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

where

$$A = e^{ik_{1z}a}\left[\cos k_{2z}b + \frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right)\sin k_{2z}b\right]$$

$$B = e^{-ik_{1z}a}\left[\frac{i}{2}\left(\frac{k_{2}}{k_{1z}} - \frac{k_{1z}}{k_{2z}}\right)\sin k_{2z}b\right]$$

$$C = e^{ik_{1z}a}\left[-\frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} - \frac{k_{1z}}{k_{2z}}\right)\sin k_{2z}b\right]$$

$$D = e^{-ik_{1z}a}\left[\cos k_{2z}b - \frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right)\sin k_{2z}b\right]$$

② TM wave (p-wave, $\vec{H}$ polarized perpendicular to yz plane)

$$A = e^{ik_{1z}a}\left[\cos k_{2z} b + \frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} + \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin k_{2z} b\right]$$

$$B = e^{-ik_{1z}a}\left[\frac{i}{2}\left(\frac{n_2^2}{n_1^2}\frac{k_{1z}}{k_{2z}} - \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin k_{2z} b\right]$$

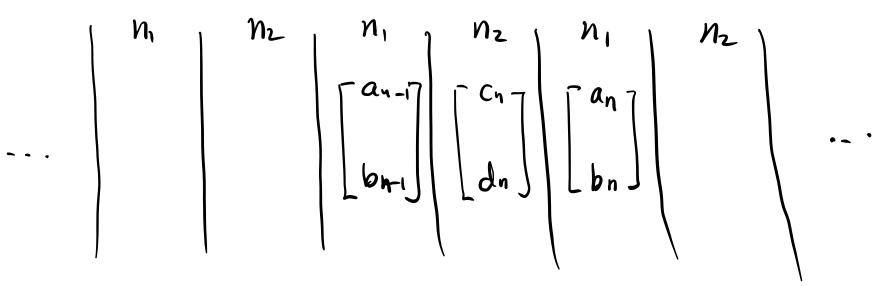
$$C = e^{ik_{1z}}\left[-\frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} - \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin k_{2z} b\right]$$

$$D = e^{-ik_{1z}}\left[\cos k_{2z} b - \frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} + \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin k_{2z} b\right]$$

Comments:

① $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ relates the complex amplitude of the plane wave in layer 1 (with index n_1) to those of the equivalent layer in the next unit cell.

i. e.



② $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = AD - BC = 1$

③ $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} a_n \\ b_n \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-n} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}^n \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

Next, solve $\vec{E}$ and $\omega(\vec{K})$!

Recall Bloch thereom.

In a periodic medium, i.e. $n(z) = n(z+\Lambda)$

For oblique incidence, the $\vec{E}$-field should be:

$$\boxed{\vec{E} = E_K(z)e^{-iKz}e^{i(\omega t - k_y z)}} \quad \text{(Bloch wave)}$$

where $\vec{E}_K(z) = \vec{E}_K(z+\Lambda)$ is a periodic function.

Note: ① $\vec{K}$ is the Bloch vector

② When $\vec{K}$ is real $\Rightarrow$ propagating wave (no loss)

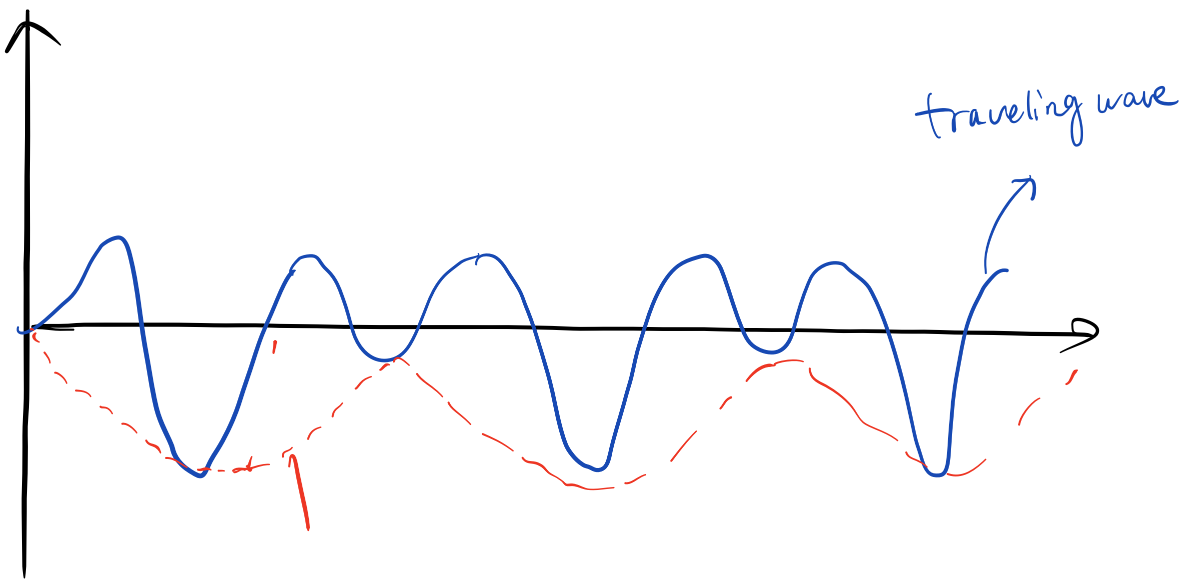
③ When $\vec{K}$ is complex $\Rightarrow$ evanescent wave

Next goal: solve $\vec{K}$ Determine $\omega(\vec{K})$!

How does a Bloch wave look like?

Bloch wave:

$$\vec{E} = \underbrace{E_k(z)\, e^{-ikz}}_{\text{spatial modulation}} \underbrace{e^{i(\omega t - k_y z)}}_{\text{plane wave}}$$



The Bloch wave requires :

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{-ik\Lambda} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}$$

Also, $\begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$.

$$\Rightarrow \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{ik\Lambda} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad e^{ik\Lambda} = \frac{1}{2}(A+D) \pm i\sqrt{\frac{1}{4}(A+D)^2 - 1}$$

$$\Rightarrow \quad \cos k\Lambda = \frac{1}{2}(A+D)$$

$$\Rightarrow \quad K(\omega, k_y) = \frac{1}{\Lambda}\cos^{-1}\left[\frac{1}{2}(A+D)\right]$$

Comments:

① When $\left|\frac{1}{2}(A+D)\right| < 1$, K is real $\Rightarrow$ propagating Bloch wave

② When $\left|\frac{1}{2}(A+D)\right| > 1$, $K = \frac{m\pi}{\Lambda} + iK_i \Rightarrow$ Decaying (Evanescent) Bloch wave.

Meaning: there is "photonic bandgap" of the periodic medium.

band edges are the regimes $\left|\frac{1}{2}(A+D)\right| = 1$.

③ Plug in A and D.

$$\cos K\Lambda = \begin{cases} \cos k_{1z}a \cos k_{2z}b - \frac{1}{2}\left(\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right)\sin k_{1z}a \sin k_{2z}b & \text{(TE)} \\ \cos k_{1z}a \cos k_{2z}b - \frac{1}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} + \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin k_{1z}a \sin k_{2z}b & \text{(TM)} \end{cases}$$

Note: $k_{1z} = \sqrt{\left(\frac{n_1\omega}{c}\right)^2 - k_y^2}$

$k_{2z} = \sqrt{\left(\frac{n_2\omega}{c}\right)^2 - k_y^2}$

Show Figure of photonic bandgap.

Note. k is a function of ω and k_y. ($K(\omega, k_y)$)

So the band diagram is 3-D

Next goal: calculated the size of photonic bandgap!

Special case: Normal incidence. ($k_y = 0$)

$$\boxed{\cos K\Lambda = \cos k_1 a \cos k_2 b - \frac{1}{2}\left(\frac{n_2}{n_1} + \frac{n_1}{n_2}\right)\sin k_1 a \sin k_2 b} \quad ①$$

where $k_1 = \left(\frac{\omega}{c}\right) n_1$, $k_2 = \left(\frac{\omega}{c}\right) n_2$

Let ω_0 be the center of the photonic bandgap, at ω_0,

$k_1 a = k_2 b = \frac{\pi}{2}$, (this condition is quater wave stack)

(i.e, $\frac{2\pi}{\lambda_1} a = \frac{2\pi}{\lambda_2} b = \frac{\pi}{2}$, $\Rightarrow a = \frac{\lambda_0}{4n_1}$, $b = \frac{\lambda_0}{4n_2}$)

$$\boxed{\cos K\Lambda = -\frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right)} \quad ②$$

When $K\Lambda = \pi + i\xi \Rightarrow$ photonic bandgap. plug in ①.

$$\xi = \log\left|\frac{n_2}{n_1}\right| \simeq \frac{2(n_2 - n_1)}{n_1 + n_2}. \quad \text{at } \omega = \omega_0$$

Define ζ as the normalized freq. deviation from the center of the first photonic bandgap ω_0, i.e.

$$\zeta = \frac{\omega - \omega_0}{c} n_1 a = \frac{\omega - \omega_0}{c} n_2 b. \quad ③$$

plug ③ and $K\Lambda = \pi + i\xi$ into ①, we have

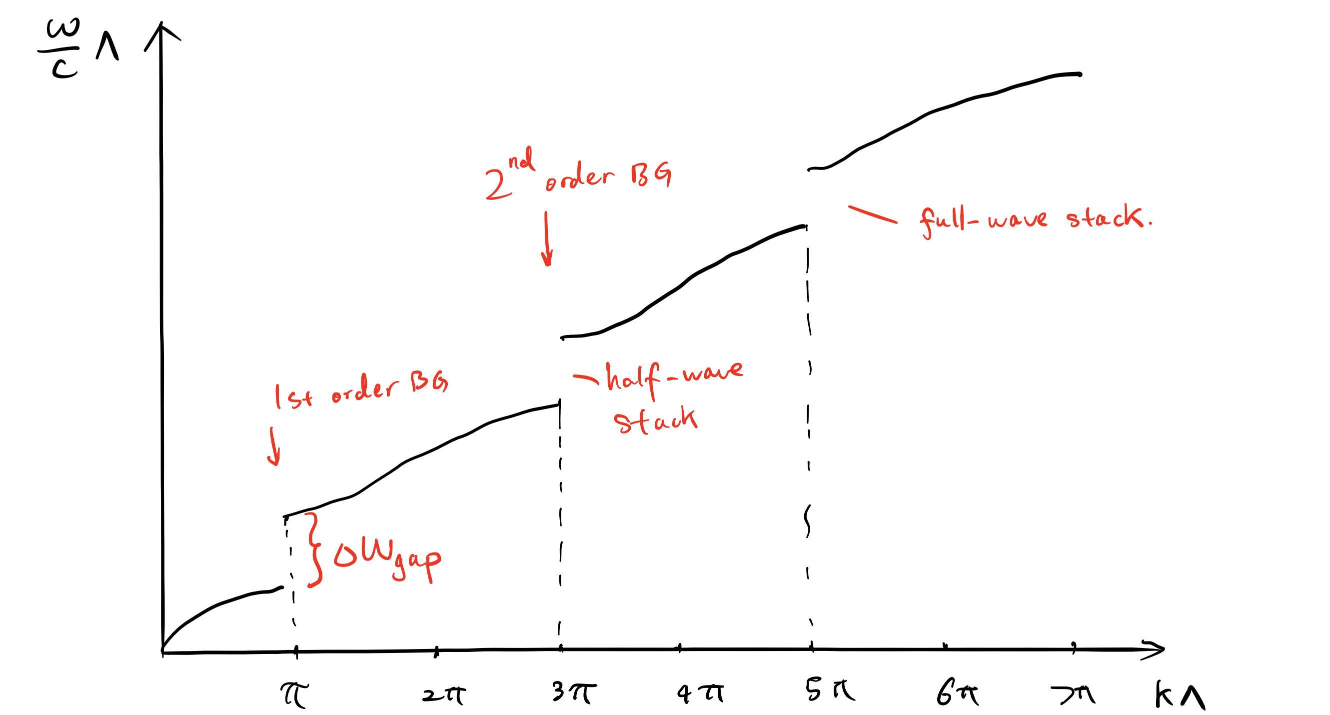
$$\cosh\xi = \frac{1}{2}\left(\frac{n_2}{n_1} + \frac{n_1}{n_2}\right)\cos^2\zeta - \sin^2\zeta$$

Bandedge: set $\xi = 0$.

$$\xi_{edge} = \frac{\omega_{edge} - \omega_0}{c} n_1 a = \frac{\omega_{edge} - \omega_0}{c} n_2 b = \pm \sin^{-1}\left(\frac{n_2 - n_1}{n_2 + n_1}\right)$$

$$\Delta\omega_{gap} = \omega_0 \frac{4}{\pi} \sin^{-1}\left|\frac{n_2 - n_1}{n_2 + n_1}\right|$$

Quarter-wave stack



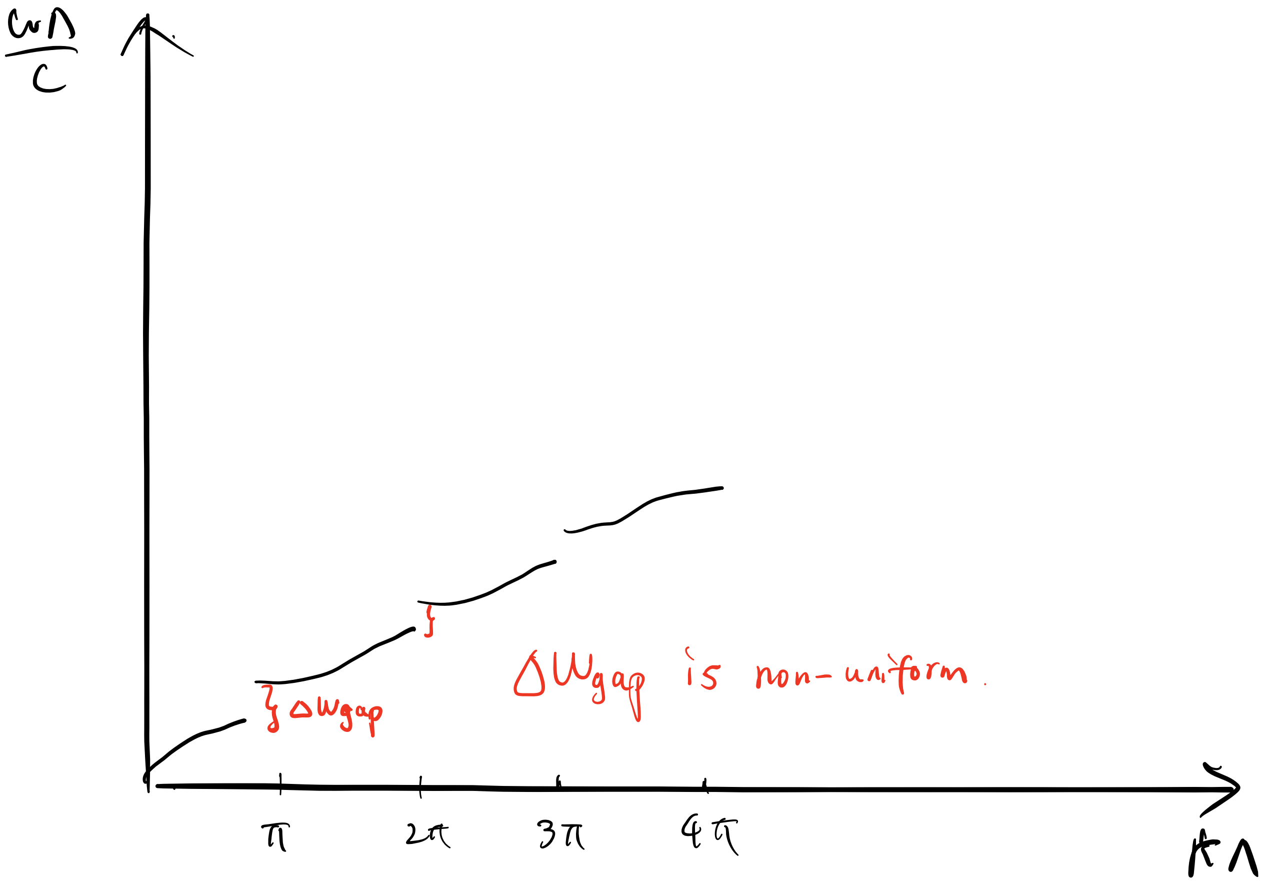
(Parameters : $n_1 = 1.45$, $n_2 = 2.65$, $a = \frac{\lambda_0}{4n_1} = 259\,nm$, $b = \frac{\lambda_0}{4n_2} = 142\,nm$, $\lambda_0 = 1.5\,\mu m$)

Comments:

① For quater-wave stack, bandgaps exist when $k\Lambda = \pi, 3\pi, 5\pi$.
even photonic bandgap at $K\Lambda = 2\pi, 4\pi, 6\pi \cdots$ vanish.

② Increasing index contrast $(n_2 - n_1)$, bandgap increases !

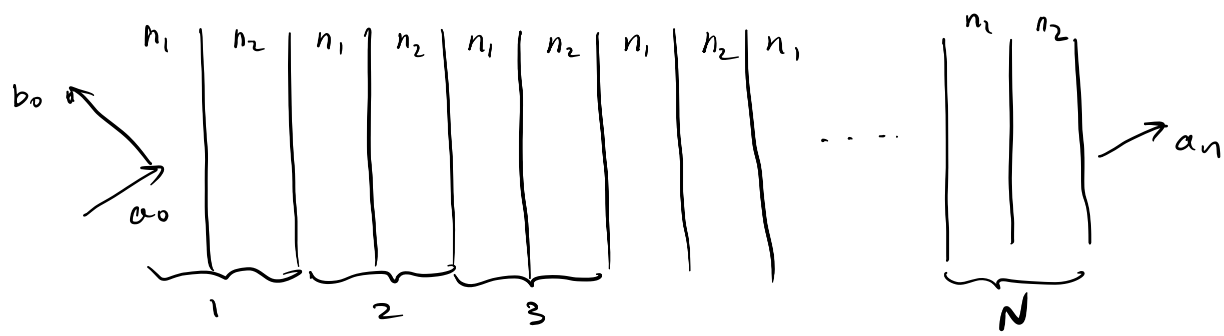
Non-quater wave stack



3. Bragg reflectors

In photonic bandgap, power flow along z axis is zero, meaning that the energy of the incident beam is totally reflected.

implication: by properly designing the periodic layered medium, it is possible to achieve near-unity reflection



reflection coefficient: $r_N = \dfrac{b_0}{a_0}$

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} AU_{N-1} - U_{N-2} & BU_{N-1} \\ CU_{N-1} & DU_{N-1} - U_{N-2} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

where $U_N = \dfrac{\sin(N+1)K\Lambda}{\sin K\Lambda}$

$$\Rightarrow \quad r_n = \frac{CU_{N-1}}{AU_{N-1} - U_{N-2}}.$$

$$|r_n|^2 = \frac{|C|^2}{|C|^2 + (\sin K\Lambda / \sin NK\Lambda)^2}, \quad \text{and } r_N = |r_N| e^{-i\varphi}$$

where $|C|^2$ is directly related to the reflectance of single period.

Comments

① When $K\Lambda = m\pi$, $m = 1, 2 \cdots$ (At the edges of photonic bands).

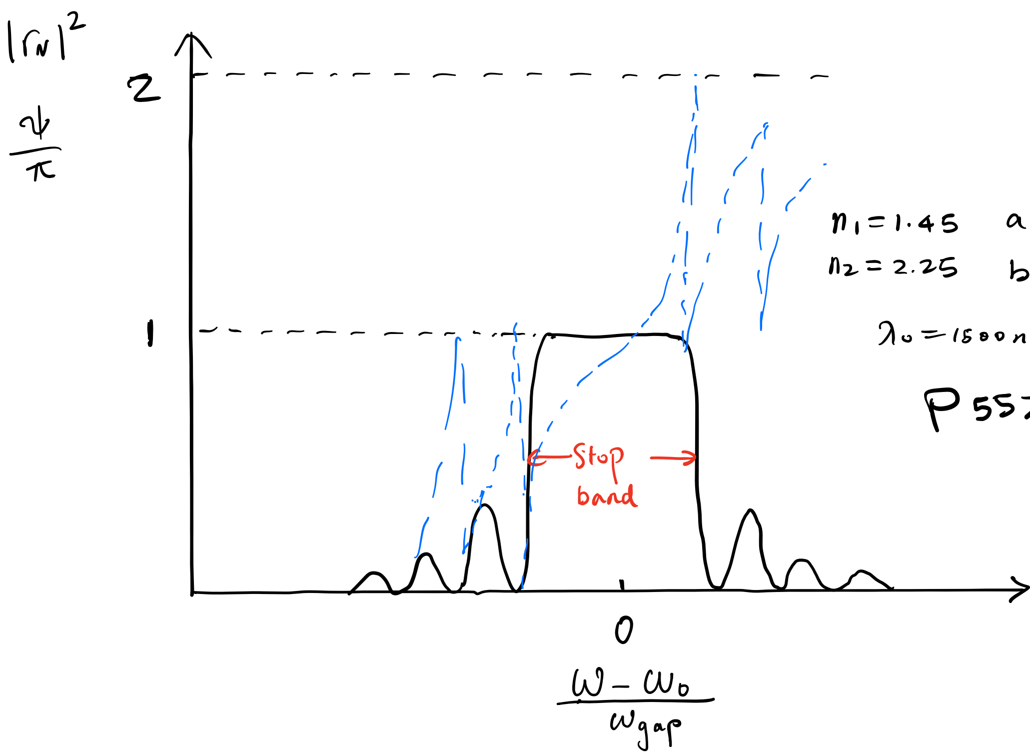
$$|r_N| = \frac{|C|^2}{|C|^2 + (1/N)^2}.$$

when $N \gg 1$, $|r_N| \simeq 1$

② Inside the photonic bandgap, $K\Lambda = m\pi + iK_i\Lambda$. (complex).

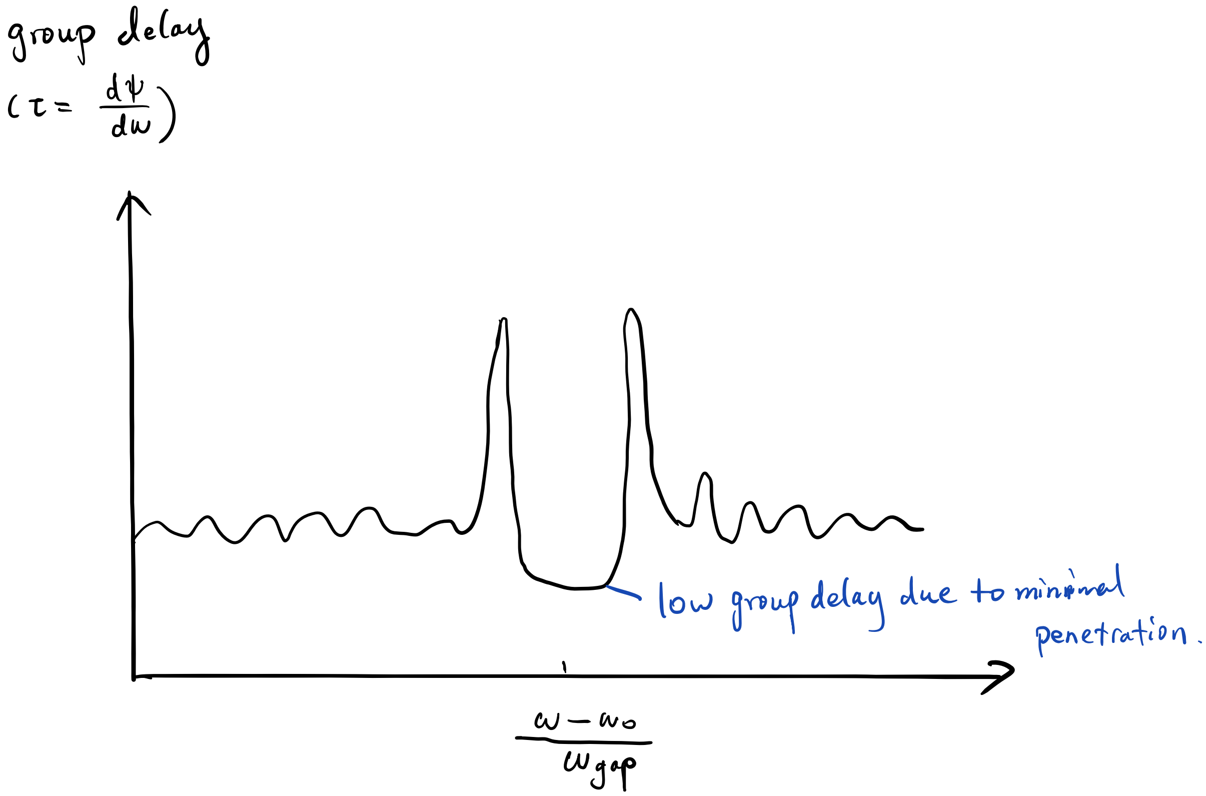
$$|r_N| = \frac{|C|^2}{|C|^2 + (\sinh K_i\Lambda / \sinh NK_i\Lambda)^2}$$

when $N \gg 1$, $(\sinh K_i\Lambda / \sinh NK_i\Lambda) \simeq 0$



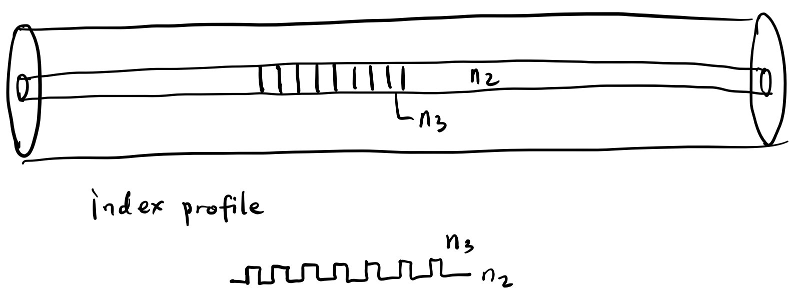
$n_1 = 1.45$ $a = 259$ nm
$n_2 = 2.25$ $b = 167$ nm
$\lambda_0 = 1500$ nm

P557 Yariv.



Applications:

① Fiber Bragg grating (FBG)



② DBR Laser

4. Transmission filter (Defect in photonic crystal)

See slides.