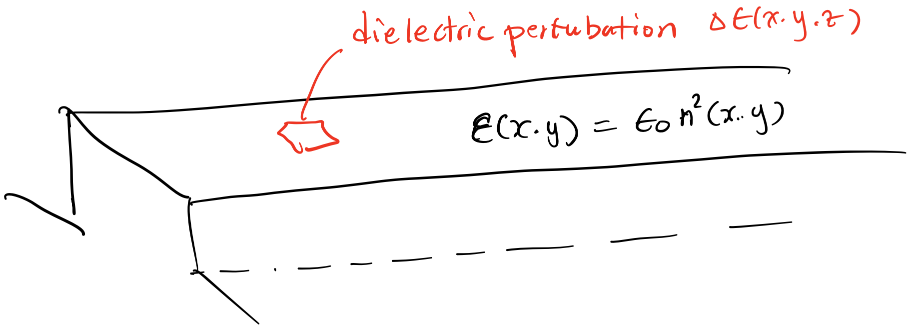
# Lecture 8. Waveguide devices

Learning objectives:

1. Mode coupling
2. Waveguide coupling (directional coupler)
3. Ring resonator and critical coupling

## 1. Mode coupling (Yariv P603)




when $\epsilon(x,y)$ is independent of $(z)$, modes are independent

but when dielectric perturbation is appied to the waveguide, (bending, surface corrugation), the modes are coupled together.

Eg. incident $TE_0$ mode, some of its power will be transfered to other modes, such as $TE_1$ $TE_2$ . . .

Specical case, $\Delta\epsilon(x.y.z) = \Delta\epsilon(x.y)$.

$$\epsilon(x.y) = \underbrace{\epsilon_a(x,y)}_{\text{dielectric const. of unperturbed WG.}} + \Delta\epsilon(x.y)$$

Let the unperturbed modes be

$$E_m = \mathcal{E}_m(x.y) \cdot e^{i(wt - \beta_m z)},$$

where $\mathcal{E}_m(x,y)$ (transverse wave functions) satisfy

$$\nabla_t^2 \mathcal{E}_m(x,y) + [\omega^2 \mu \epsilon_a(x,y) - \beta_m^2]\mathcal{E}_m(x,y) = 0 \quad ①$$

Apply the perturbation, we have

$$[\nabla_t^2 + \omega^2\mu\epsilon_a(x,y) + \omega^2\mu\Delta\epsilon(x,y)](\mathcal{E}_m + \delta\mathcal{E}_m) = (\beta_m^2 + \delta\beta_m^2)(\mathcal{E}_m + \delta\mathcal{E}_m)$$

neglect the second order term $\Delta\epsilon \cdot \delta\mathcal{E}_m$ and $\delta\beta_m^2 \delta\mathcal{E}_m$, use eq. ①, we have

$$[\nabla_t^2 + \omega^2\mu\epsilon_a(x,y)]\delta\mathcal{E}_m + \omega^2\mu\Delta\epsilon\mathcal{E}_m = \beta_m^2\delta\mathcal{E}_m + \delta\beta_m^2\mathcal{E}_m \quad ②$$

To solve ②, we expand $\delta\mathcal{E}_m(x,y) = \sum_n a_{mn}\mathcal{E}_n(x,y)$ plug in ②. we have

$$\sum_n a_{mn}(\beta_n^2 - \beta_m^2)\mathcal{E}_n(x,y) = (\delta\beta_m^2 - \omega^2\mu\Delta\epsilon)\mathcal{E}_m(x,y) \quad ③$$

Scalar-multiply ③ by $\mathcal{E}_m^*$, integral over $x,y$, use the orthogonal property.

i.e. $\int_A \mathcal{E}_m^*(x,y)\,\mathcal{E}_n(x,y)\,dx\,dy = 0$,

the left term is gone.

We get.

$$\iint \mathcal{E}_m^* \cdot (\delta\beta_m^2 - \omega^2\mu \cdot \Delta\epsilon)\, \mathcal{E}_m(x,y) = 0$$

$$\Rightarrow \quad \delta\beta_m^2 = \frac{\iint \mathcal{E}_m^* \cdot \omega^2 \mu\, \Delta\epsilon\, \mathcal{E}_m \, dx\, dy}{\iint \mathcal{E}_m^* \mathcal{E}_m \, dx\, dy}$$

Using the orthogonal relation: $\frac{\beta_m}{2\omega\mu} \iint \mathcal{E}_m^* \mathcal{E}_n^* \, dx \cdot dy = \delta_{mn}$.

$$\Rightarrow \quad \delta\beta_m = \frac{\omega}{4} \iint \mathcal{E}_m^* \cdot \Delta\epsilon\, \mathcal{E}_m \, dx \cdot dy .$$

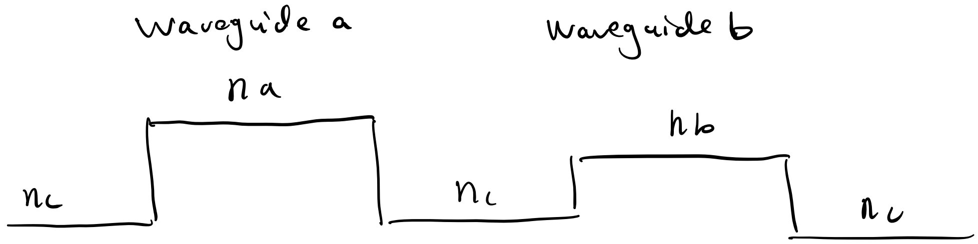
$$a_{mn} = \frac{\omega \beta_n}{2(\beta_m^2 - \beta_n^2)} \iint \mathcal{E}_n^* \cdot \Delta\epsilon(x,y)\, \mathcal{E}_m \, dx \cdot dy$$

$$a_{mm} = -\frac{\omega}{8\beta_m} \iint \mathcal{E}_m^* \, \Delta\epsilon(x,y) \cdot \mathcal{E}_m \, dx\, dy$$

Define $K_{nm} = \delta\beta_m = \frac{\omega}{4} \iint \mathcal{E}_n^* \, \Delta\epsilon\, \mathcal{E}_m \, dx \cdot dy$.

as the "coupling coefficient" between modes.

## 2. Waveguide coupling – Directional coupler




Mode:

$$\mathcal{E}_a(x,y)\,e^{i(\omega t-\beta_a z)} \qquad \mathcal{E}_b(x,y)\,e^{i(\omega t-\beta_b z)}$$

When they are closer (not too close!),
the general wave propagation in the coupled waveguide is

$$\vec{E}(x,y,z,t) = A(z)\,\mathcal{E}_a(x,y)\,e^{i(\omega t-\beta_a z)} + B(z)\,\mathcal{E}_a\,e^{i(\omega t-\beta_b z)}$$

Note:

① if distance between a. b is infinite,
$A(z)$, $B(z)$ do not depend on $z$,

Index distribution $n^2(x,y)$

$$n^2(x,y) = \begin{cases} n_a^2 & \text{core } a \\ n_b^2 & \text{core } b \\ n_c^2 & \text{elsewhere} \end{cases}$$

define $$\Delta n_a^2(x,y) = \begin{cases} n_a^2 - n_c^2 & \text{core } a \\ 0 & \text{elsewhere} \end{cases}$$

$$\Delta n_b^2(x,y) = \begin{cases} n_b^2 - n_c^2 & \text{core } b \\ 0 & \text{elsewhere} \end{cases}$$

$$n_s^2(x,y) = n_c^2$$

The index profile: $n^2(x,y) = n_s^2(x,y) + \Delta n_a^2(x,y) + \Delta n_b^2(x,y)$

So, the wave equation for the composite waveguide:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c} \left[ n_s^2(x,y) + \Delta n_a^2(x,y) + \Delta n_b^2(x,y) \right] \right) \vec{E} = 0$$

where $\vec{E} = A(z)\,\mathcal{E}_a(x,y)\,e^{i(\omega t - \beta_a z)} + B(z)\,\mathcal{E}_b(x,y)\,e^{i(\omega t - \beta_b z)}$

Goal: solve $A(z)$, $B(z)$!

Assuming slow variation of mode amplitudes over $z$,

we get

$$-2i\beta_a \frac{dA}{dz} \mathcal{E}_a e^{i(\omega t-\beta_a z)} - 2i\beta_b \frac{dB}{dz} \mathcal{E}_b e^{i(\omega t-\beta_b z)}$$

$$= -\frac{\omega^2}{c^2} \Delta n_b^2(x.y) A \mathcal{E}_a e^{i(\omega t-\beta_a z)} - \frac{\omega^2}{c^2} \Delta n_a^2(x.y) B \mathcal{E}_b e^{i(\omega t-\beta z)}$$

Take scalar product with $\mathcal{E}_a^*(x.y)$ $\mathcal{E}_b^*(x.y)$ and integrate over $x.y$ plane, we get

$$\left.\begin{aligned} \frac{dA}{dz} &= -i\kappa_{ab} B e^{i(\beta_a-\beta_b)z} - i\kappa_{aa} A . \\ \frac{dB}{dz} &= -i\kappa_{ba} A e^{-i(\beta_a-\beta_b)z} - i\kappa_{bb} B \end{aligned}\right\} \quad (4)$$

Coupled mode eq. (CME)

where $\kappa_{ab} = \frac{\omega}{4}\epsilon_0 \iint \mathcal{E}_a^* \cdot \Delta n_a^2(x.y)\, \mathcal{E}_b \, dx\, dy$

$\kappa_{ba} = \frac{\omega}{4}\epsilon_0 \iint \mathcal{E}_b^* \, \Delta n_b^2(x.y)\, \mathcal{E}_a \, dx . dy$

$\kappa_{aa} = \frac{\omega}{4}\epsilon_0 \iint \mathcal{E}_a^* \, \Delta n_b^2(x.y)\, \mathcal{E}_a \, dx\, dy$

$\kappa_{bb} = \frac{\omega}{4}\epsilon_0 \iint \mathcal{E}_b^* \, \Delta n_a^2(x.y)\, \mathcal{E}_b \, dx\, dy$

## Comments:

1. $\kappa_{ab}$, $\kappa_{ba}$ are exchange coupling between 2 waveguides
2. $\kappa_{aa}$, $\kappa_{bb}$ result from pertubation to one waveguide due to the presence of another

③ $\kappa_{ab} = \kappa_{ba}^*$, ensure the conservation of energy.

If assuming

$$E(x,y,z,t) = A(z)\,\mathcal{E}_a\, e^{i(\omega t - (\beta_a + \kappa_{aa})z)} + B(z)\,\mathcal{E}_b\, e^{i(\omega t - (\beta_b + \kappa_{bb})z)}$$

CME ④ reduces to

$$\begin{cases} \dfrac{dA}{dz} = -i\kappa_{ab} B e^{i2\delta z} \\ \dfrac{dB}{dz} = -i\kappa_{ba} A e^{-i2\delta z}, \end{cases} \quad 2\delta = (\beta_a + \kappa_{aa}) - (\beta_b + \kappa_{bb}) \qquad ⑤$$

Assuming $\kappa_{ab} = \kappa_{ba} = \kappa$, $s = \sqrt{\kappa^2 + \delta^2}$, solution of ⑤ is

$$\begin{cases} A(z) = A_0 e^{i\delta z}\left(\cos sz - i\dfrac{\delta}{s}\sin sz\right) \\ B(z) = -iA_0 e^{-i\delta z}\dfrac{\kappa}{s}\sin sz, \end{cases}$$

In terms of power, $P_a(z) = |A(z)|^2$, $P_b(z) = |B(z)|^2$

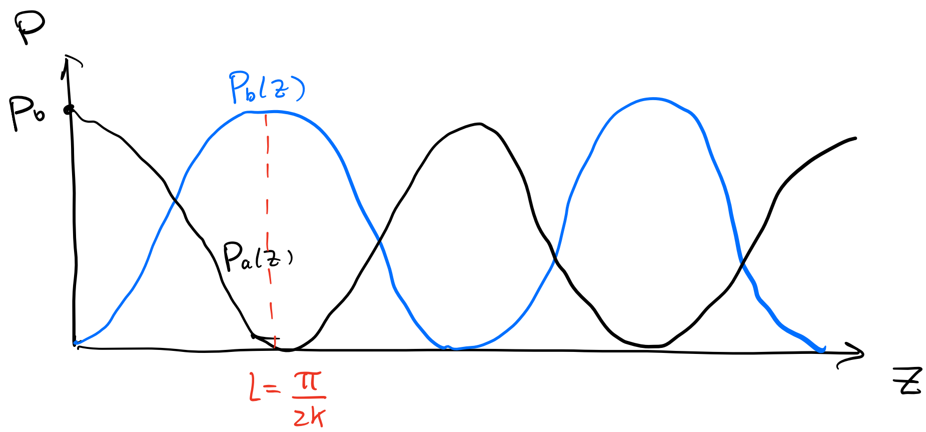
$$\begin{cases} P_a(z) = P_0 - P_b(z) & ① \\ P_b(z) = P_0 \dfrac{\kappa^2}{\kappa^2 + \delta^2}\sin^2\sqrt{\kappa^2 + \delta^2}\, z & ② \end{cases}$$

where $P_0 = |A(0)|^2 = A_0^2$ is the input power at $z = 0$

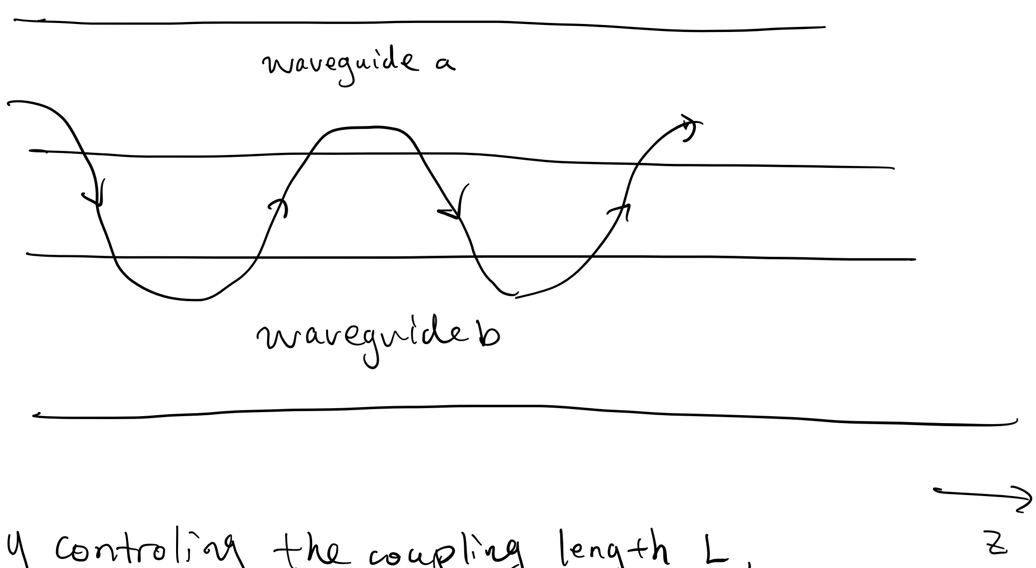
## Comments

1) when $\delta = 0$, ($\beta_a = \beta_b$), ② becomes

$$\begin{cases} P_a(z) = P_0 - P_b(z) \\ P_b(z) = P_0 \sin^2 kz , \end{cases}$$




Directional coupler:

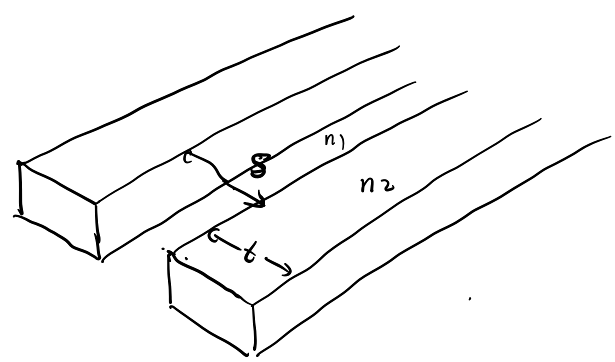



by controling the coupling length L,

we can control $P_a(z)/P_b(z)$

2) For $\delta \neq 0$, the maximum fraction of power that can be transferred is $\dfrac{\kappa^2}{\kappa^2+\delta^2}$

For two identical channel waveguides




If modes are well confined ($t \gg 2q$)

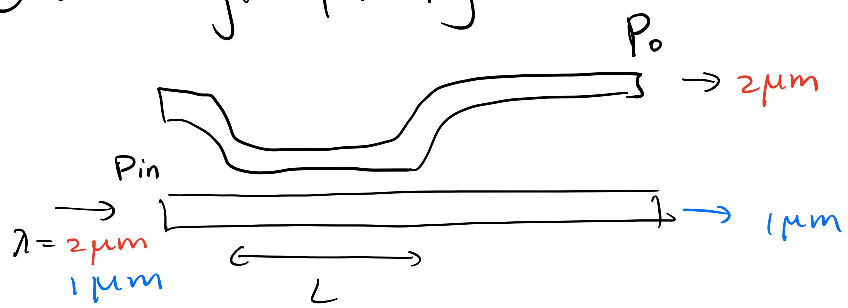
$$\kappa = \frac{2h^2 q e^{-qs}}{\beta t (h^2+q^2)} \left(\frac{2\pi}{\lambda}\right)^2 (n_2^2 - n_1^2)$$

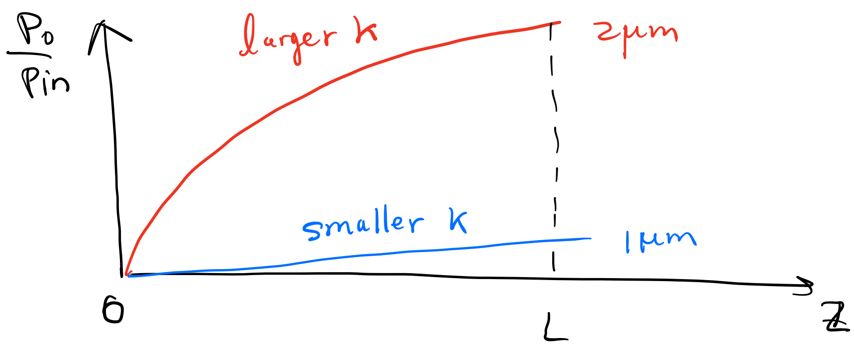
At $\lambda = 1\,\mu m$, $t, s \sim 3\,\mu m$, $\Delta n = 5\times10^{-3}$,

$\kappa = 5\,cm^{-1}$, coupling length $L \sim \kappa^{-1} = 2mm$.

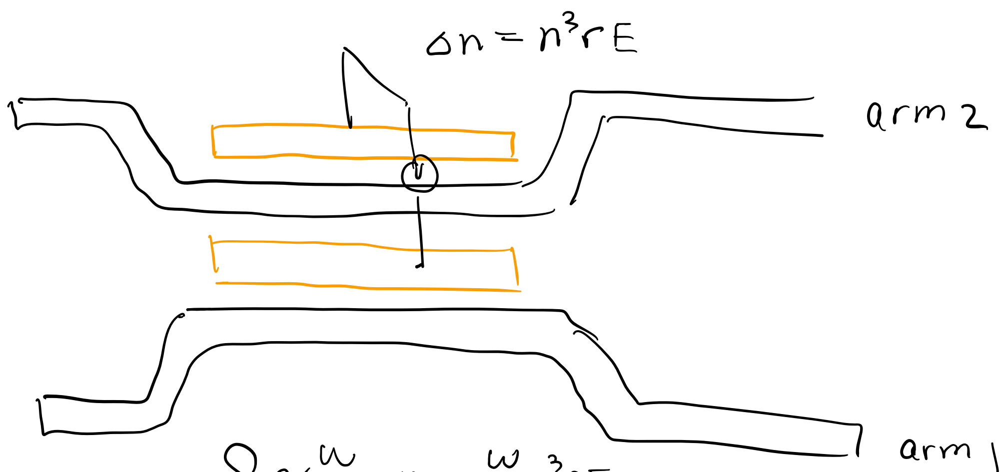
# Applications

① Wavelength filtering

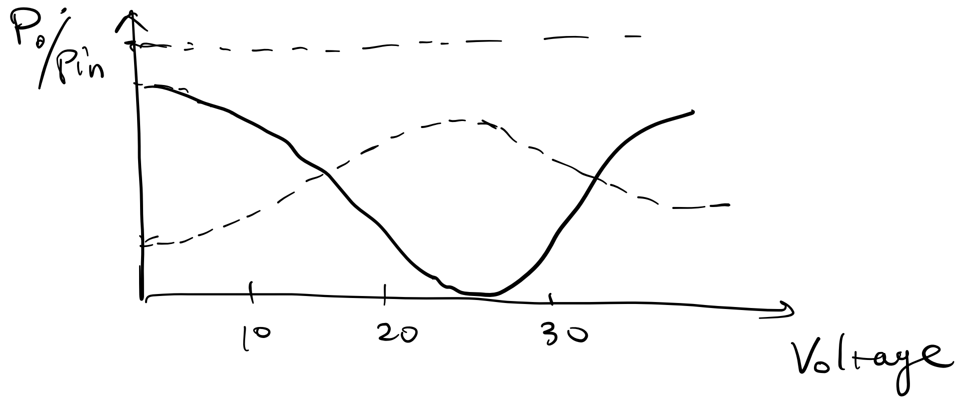



② Electro-optic swith

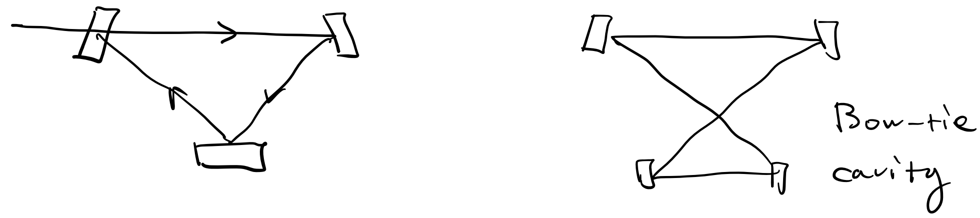



$$\delta \sim \frac{\omega}{c} \Delta n = \frac{\omega}{c} n^3 r E$$

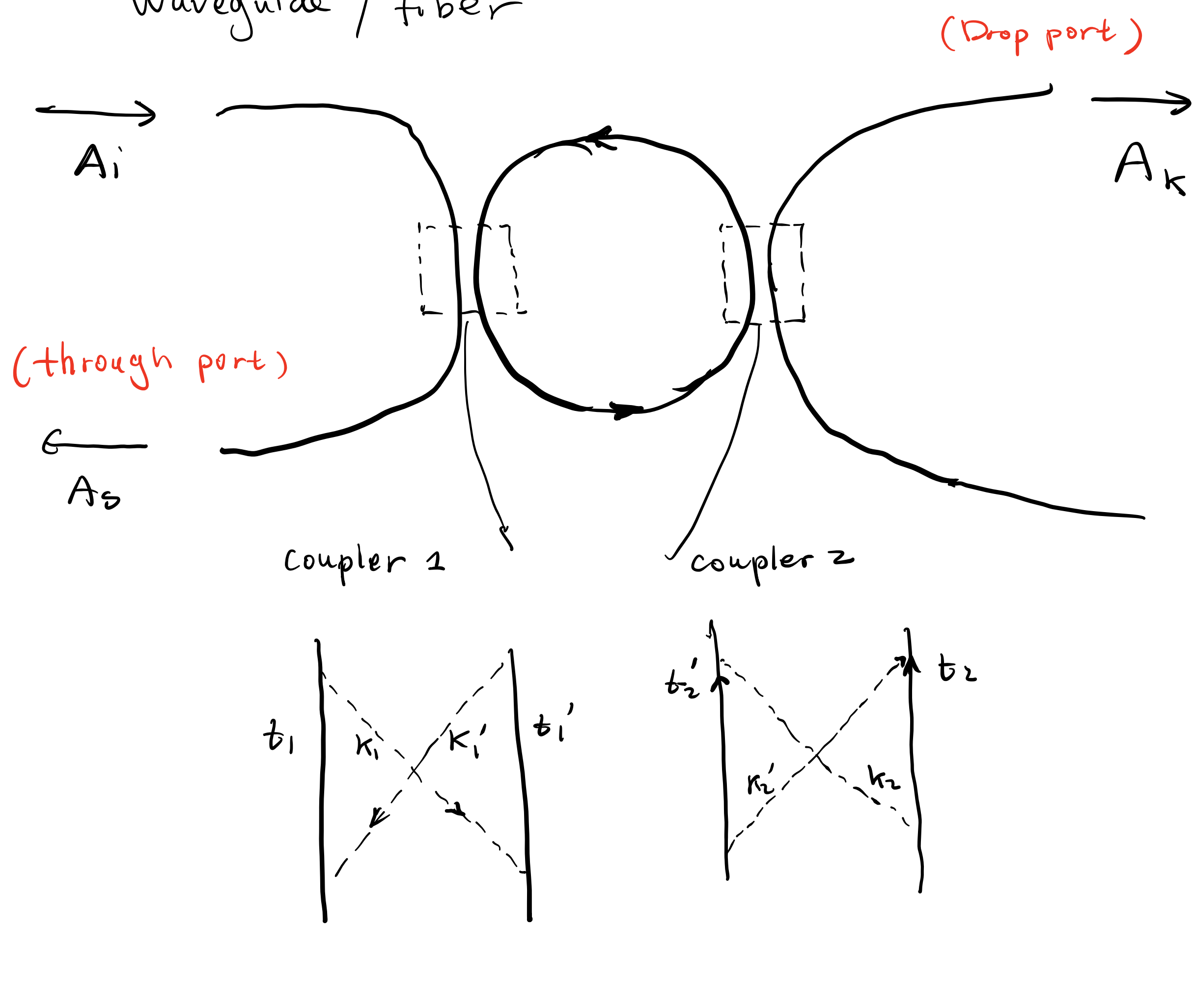

$P_o/P_{in}$
10
20
30
Voltage

## 3. Ring resonators and critical coupling (Yariv, P185)

Free space ring resonators:




Waveguide / fiber



Define ① straight-through coupling coefficient $t_1, t_1', t_2\ t_2'$

② Cross-coupling coefficient $\kappa_1, \kappa_1'\ \kappa_2\ \kappa_2'$

Model the ring as Fabry-Perot Etalon, Then,

$$A_s = t_1 A_i + \kappa_1 \kappa_1' t_2' e^{-i\delta} A_i + \kappa_1 \kappa_1' t_2' t_1' t_2' e^{-2i\delta} + \cdots$$

$$= \left\{ t_1 + \kappa_1 \kappa_1' t_2' e^{-i\delta} \left[ 1 + t_1' t_2' e^{-i\delta} + (t_1' t_2' e^{-i\delta})^2 + \cdots \right] \right\} A_i$$

$$A_K = \kappa_1 \kappa_2' e^{-i\delta/2} \left[ 1 + t_1' t_2' e^{-i\delta} + (t_1' t_2' e^{-i\delta})^2 + \cdots \right] A_i$$

where $\delta = \dfrac{2\pi n_{eff}\, d}{\lambda}$ ← circumference of the ring.

Summing up the geometric series.

$$\sigma = \frac{A_s}{A_i} = \frac{t_1 + (\kappa_1 \kappa_1' - t_1 t_1') t_2' e^{-i\delta}}{1 - t_1' t_2' e^{-i\delta}}$$

$$\chi = \frac{A_K}{A_i} = \frac{\kappa_1 \kappa_2' e^{-i\frac{\delta}{2}}}{1 - t_1' t_2' e^{-i\delta}}$$

When there is loss due to scattering, absorption and bending.

$$\sigma = \frac{A_s}{A_i} = \frac{t_1 + (k_1 k_1' - t_1 t_1') t_2' e^{-i\delta} e^{-\alpha d}}{1 - t_1' t_2' e^{-i\delta} e^{-\alpha d}}$$ (T from input to through)

$$\chi = \frac{A_k}{A_i} = \frac{k_1 k_2 e^{-i\frac{\delta}{2}} e^{-\alpha d}}{1 - t_1' t_2' e^{-i\delta} e^{-\alpha d}}$$ (T from input to drop)

which is quite similar to F-P etalon (cavity).

**Q**: How to detemine $(t_1, t_1', t_2 t_2')$ and $(k_1 k_1', k_2, k_2')$?

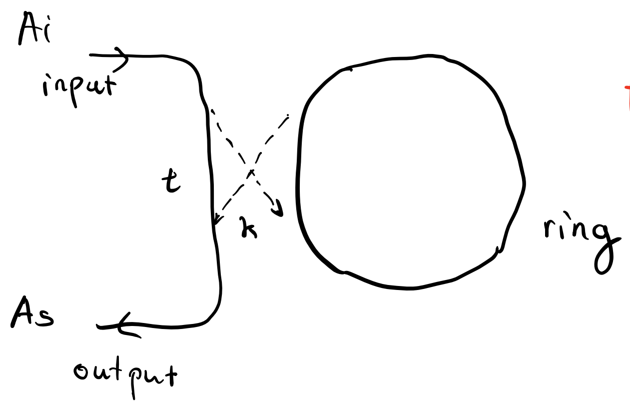
**A**: They are not independent of each other. They are related by reciprocity, conservation of energy and time-reversal symmetry.

i.e. $$\begin{cases} t_1^2 + k_1^2 = t_1'^2 + k_1'^2 = 1 \\ t_1 t_1' - k_1 k_1' = -1 \end{cases}$$ (Yariv P186~188)

Then we have:

$$\sigma = \frac{t_1 + t_2' e^{-i\delta} e^{-\alpha d}}{1 - t_1' t_2' e^{-i\delta} e^{-\alpha d}}$$

Assuming $k_2 = 0$, $t_2 = -1$, we have




$$\boxed{|t|^2 + |k|^2 = 1}$$

$$\sigma = \frac{A_s}{A_i} = \frac{t_1 - a e^{-i\delta}}{1 + t_1' a e^{-i\delta}} = \frac{t - a e^{-i\delta}}{1 - t^* a e^{-i\delta}}$$

where $t_1' = -t_1^* = -t^*$, $a = \exp(-\alpha d)$

$$|\sigma|^2 = \left|\frac{A_s}{A_i}\right|^2 = \frac{a^2 + t^2 - 2at\cos\delta}{1 + a^2 t^2 - 2at\cos\delta}$$

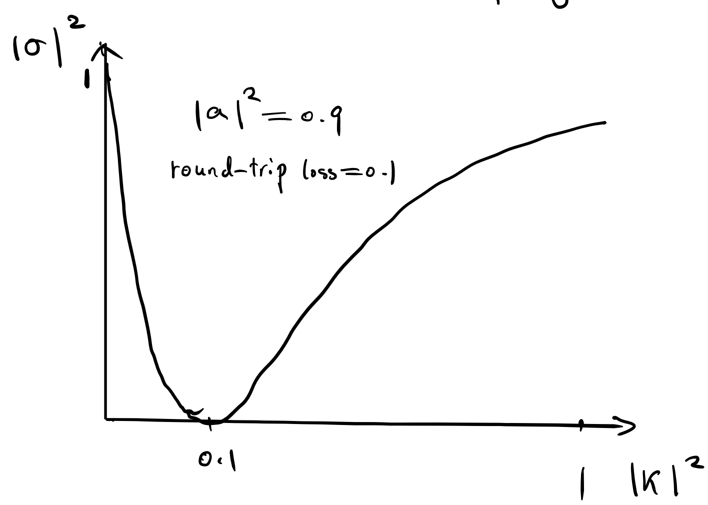
When at resonance ( $\delta = 2m\pi$, $m = 1, 2 \ldots$ )

$$\boxed{|\sigma|^2 = \left|\frac{A_s}{A_i}\right|^2 = \frac{(a-t)^2}{(1-at)^2}}$$

## Comments:

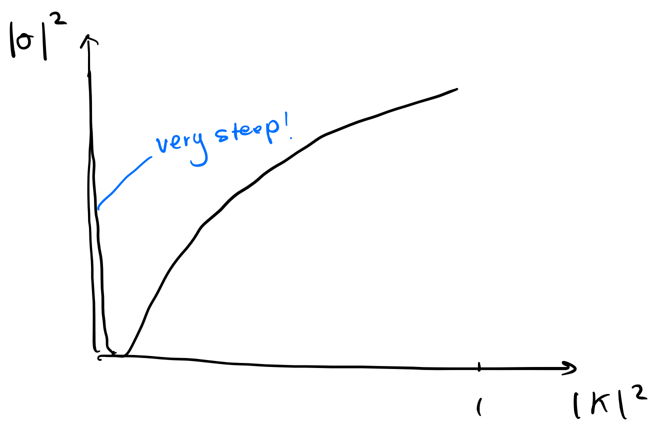
① $|\sigma|^2 = 0$ when $a = t = \sqrt{1-|\kappa|^2}$, i.e. $|\kappa|^2 = 1 - |a|^2$ = round-trip loss

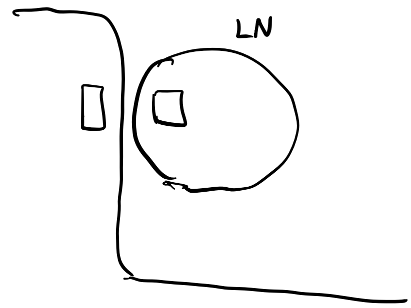
This is known as critical coupling condition




② For high Finess, high Q-resonators, $|a| \sim 1$.

round-trip loss $= 1 - |a| \simeq 0$.




A small modulation of $|\kappa|$ will modulate the transmission from zero to unity. → Applications, high modulation depth modulators.