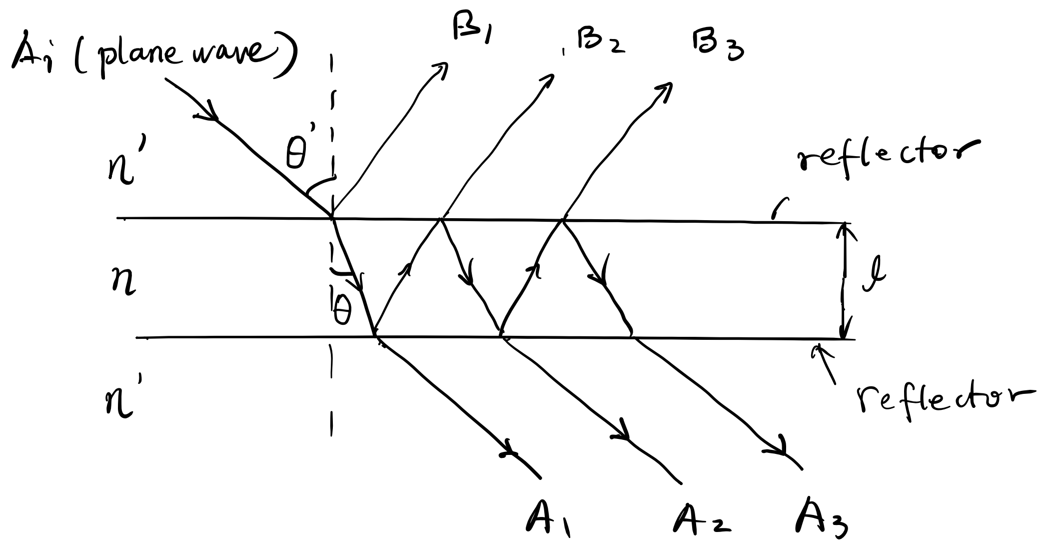
Lecture 5: Thin-film optical devices

Learning objectives:

1. Symmetric Fabry-Perot etalon
2. Asymmetric " " "
3. Finesse and quality (Q-) factor
4. Intracavity field
5. Multicavity etalons and transfer matrix

1. Symmetric Fabry-Perot etalon. (P161. Yariv)



Define : r : Fresnel reflection coefficient of the interface

t : transmission coefficient for waves incident from n' to n

r' : reflection .. .. n to n'

t' transmission .. .. .. n to n'

Reflected waves:

$$B_1 = rA_i$$

$$B_2 = tt'r'A_i e^{-i\delta} \text{ — phase shift}$$

$$B_3 = tt'r'^3 A_i e^{-i2\delta}$$

...

Transmitted waves:

$$A_1 = tt'A_i e^{-i\delta/2}$$

$$A_2 = tt'r'^2 e^{-i\delta} A_i e^{-\frac{i\delta}{2}}$$

$$A_3 = tt'r'^4 e^{-2i\delta} A_i e^{-\frac{i\delta}{2}}$$

...

The total reflected wave:

$$A_r = \left[r + tt'r' e^{-i\delta} \left(1 + r'^2 e^{-i\delta} + r'^4 e^{-2i\delta} + \cdots \right) \right] A_i$$

$$\overset{r' = -r}{=} \frac{(1 - e^{-i\delta}) r}{1 - r^2 e^{-i\delta}} A_i$$

$$\overset{R = r^2 = r'^2}{=} \frac{(1 - e^{-i\delta}) \sqrt{R}}{1 - R e^{-i\delta}} A_i$$

Total transmitted wave

$$A_t = A_i tt' \left(1 + r'^2 e^{-i\delta} + r'^4 e^{-2i\delta} + \cdots \right) e^{-i\frac{\delta}{2}}$$

$$\underline{\underline{\overset{T = tt',\ r^2 + tt'^2 = 1}{=}}} \frac{T e^{-i\frac{\delta}{2}}}{1 - R e^{-i\delta}} A_i$$

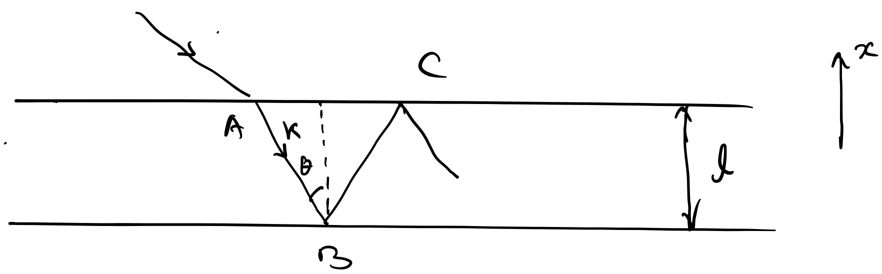
Intensity ratio:

$$\frac{I_r}{I_i} = \frac{A_r A_r^*}{A_i A_i^*} = \frac{4R \sin^2\left(\frac{\delta}{2}\right)}{(1-R)^2 + 4R \sin^2(\delta/2)}$$

$$\frac{I_t}{I_i} = \frac{A_t \cdot A_t^*}{A_i \cdot A_i^*} = \frac{(1-R)^2}{(1-R)^2 + 4R \sin^2(\delta/2)}$$

Clearly, $I_r + I_t = I_i$, power is conserved.

Let's look into the phase shift (δ)



One round trip:

$$\Delta L = AB + BC = 2l \cdot \cos\theta$$

$$\Rightarrow \delta = k \cdot \Delta L = n \cdot k_0 \Delta L = \frac{2\pi n}{\lambda} \cdot 2l\cos\theta = \frac{4\pi n l}{\lambda}\cos\theta$$

$$= \underline{2k_x \cdot l}$$

<u>Summary:</u>

Transmission $\frac{I_t}{I_i} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2\left(\frac{\delta}{2}\right)}$ becomes unity

when $\frac{\delta}{2} = m\pi$,

i.e. $\delta = \frac{4\pi n l}{\lambda}\cos\theta = 2m\pi$, $m = 1, 2 \ldots$

translate λ into frequency ν using $\nu = \frac{c}{\lambda}$,

$$\nu_m = m\frac{c}{2nl\cos\theta}, \quad m = 1, 2 \ldots$$

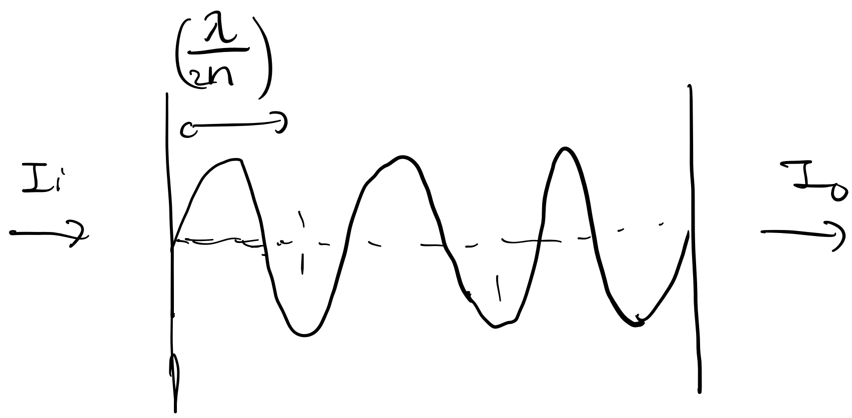
or $l = m\left(\frac{\lambda}{2n\cos\theta}\right)$, $m = 1, 2 \ldots$

For nomal incidence, $\theta = 0°$

Resonance condition:

$$\nu_m = m \cdot \frac{c}{2nl},$$

$$l = m\left(\frac{\lambda}{2n}\right).$$

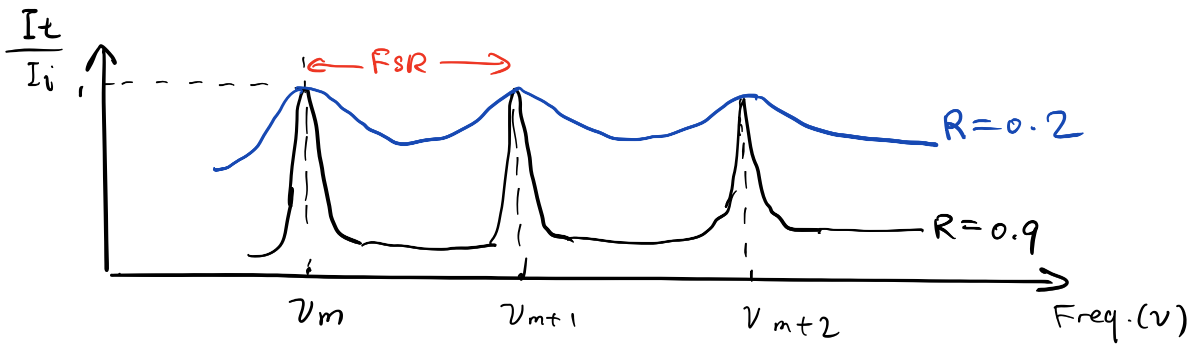


Physical meaning:

resonance happens when l is an integral number of half-wavelength.

② The transmission is unity at discrete ν_m,

$$\boxed{\Delta\nu = \nu_{m+1} - \nu_m = \frac{c}{2n\,l\cos\theta}}$$ → Free spectral range (FSR)



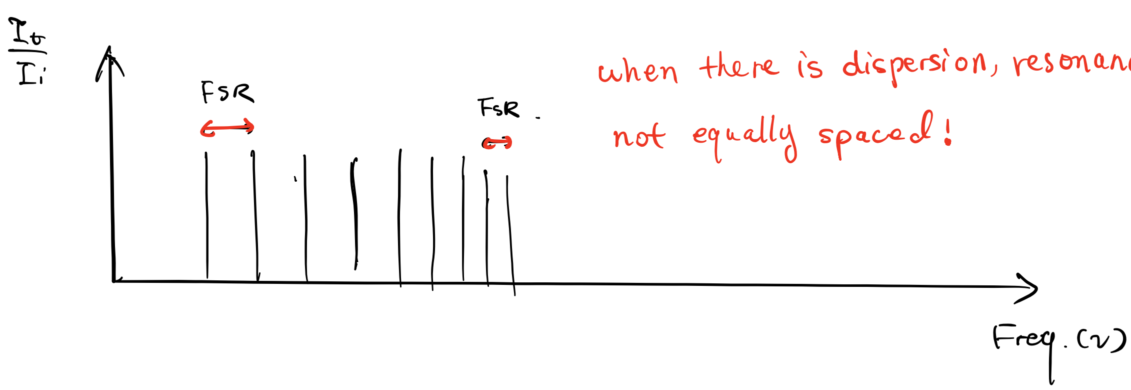
larger l leads to smaller FSR

③ ignore dispersion, FSR is a constant,

with dispersion,

$$FSR = \Delta\nu = \nu_{m+1} - \nu_m = \frac{c}{2n\;l\cos\theta} = \frac{v_g}{2l\cos\theta_i}$$

Physical meaning:



when there is dispersion, resonances not equally spaced!

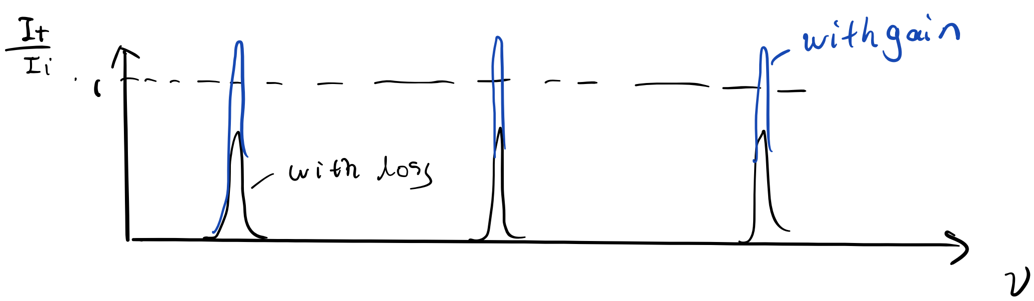
④ When there is gain/loss, maximum transmission is not unity.
(more usual cases)

Define $G = \exp(-\alpha l)$ as the intensity gain (or loss) per pass. $\begin{cases} \alpha > 0 & \text{loss} \\ \alpha < 0, & \text{gain} \end{cases}$

$$\frac{I_t}{I_i} = \frac{A_t A_t^*}{A_i A_i^*} = \frac{(1-R)^2 G}{(1-GR)^2 + 4GR\sin^2\left(\frac{\delta}{2}\right)} \quad \text{(homework)}$$

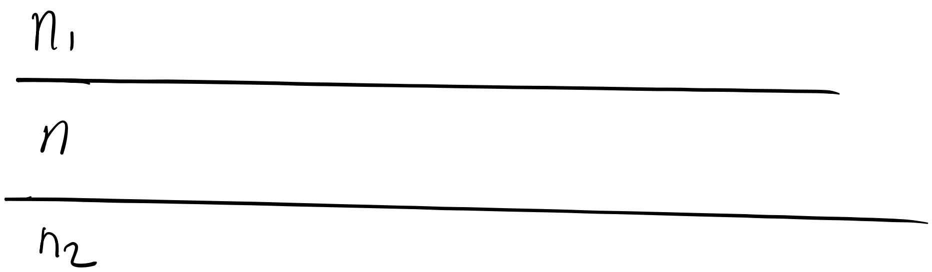
The maxium transimssion

$$\left(\frac{I_t}{I_i}\right)_{max} = \frac{(1-R)^2 G}{(1-GR)^2}$$



2. Asymmetric Fabry-Perot etalons. (P167, Yariv)

generic case: $n_1 \neq n_2$



$$r = \frac{A_r}{A_i} = \frac{r_1 + (t_1 t_1' - r_1 r_1') r_2' e^{-i\delta}}{1 - r_1' r_2' e^{-i\delta}}$$

$$t = \frac{A_t}{A_i} = \frac{t_1 t_2' e^{-i\frac{\delta}{2}}}{1 - r_1' r_2' e^{-i\delta}}$$

Use the Stokes relationships $\left(\begin{matrix} tt'^{*} + rr^{*} = 1 \\ tr'^{*} + rt^{*} = 0 \end{matrix} \right)$ (HW)

$$T_{etalon} = \frac{A_t A_t^{*}}{A_i A_i^{*}} = \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1}\sqrt{R_2})^2 + 4\sqrt{R_1}\sqrt{R_2}\sin^2(\phi)}$$

$$R_{etalon} = \frac{A_r A_r^{*}}{A_i A_i^{*}} = \frac{(\sqrt{R_1}-\sqrt{R_2})^2 + 4\sqrt{R_1}\sqrt{R_2}\sin^2(\phi)}{(1-\sqrt{R_1}\sqrt{R_2})^2 + 4\sqrt{R_1}\sqrt{R_2}\sin^2(\phi)}$$

R_1, R_2 are the reflectivity of interfaces, $2\phi = \delta + \rho_1' + \rho_2'$

ρ_1', ρ_2' are the phase shifts of reflections from interface 1 and interface 2, i.e. $r_1' = |r_1'| e^{-i\rho_1'}$, $r_2' = |r_2'| e^{-i\rho_2'}$

Summary

① For lossless mirrors, $R_1 + T_1 = 1$, $R_2 + T_2 = 1$,

$\Rightarrow T_{etalon} + R_{etalon} = 1$

② When $R_1 \neq R_2$. T_{etalon} cannot go to 1

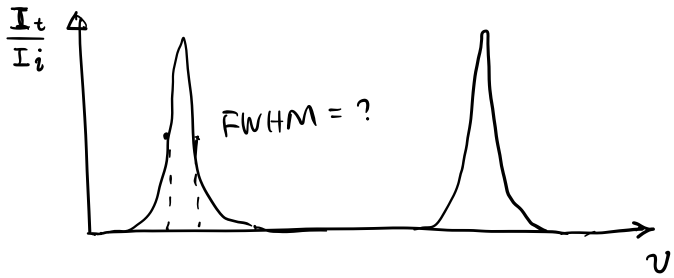
only when $R_1 = R_2$ (symmetric F-P etalon), $T = 1$ can occur!

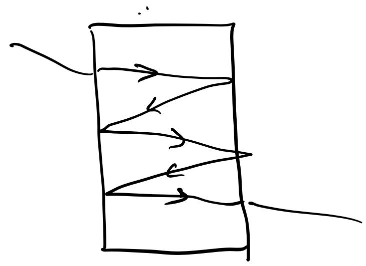
③ With gain/loss, $G = \exp(-\alpha L)$

$$(T_{etalon})_{max} = \frac{(1-\sqrt{R_1})(1-\sqrt{R_2})\,G}{(1-\sqrt{R_1}\sqrt{R_2}G)^2}$$

3. Finesse, photon lifetime, Quality factor

① Finesse.





$$\frac{I_t}{I_i} = \frac{A_t A_t^*}{A_i A_i^*} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\delta/2)} = \frac{1}{2}$$

$$\Rightarrow \sin^2\left(\frac{\delta - 2m\pi}{2}\right) = \frac{(1-R)^2}{4R}$$

Assuming the linewidth is small, $\delta - 2m\pi \ll \pi$,

$$\Delta\nu_{1/2} = \frac{c}{2\pi n l\cos\theta}\left(\delta_{1/2} - 2m\pi\right) \simeq \frac{c}{2\pi n l\cos\theta}\,\frac{1-R}{\sqrt{R}}$$

$$= \frac{FSR}{\frac{\pi\sqrt{R}}{1-R}}$$

$$= \frac{FSR}{F}$$

Define: $\mathcal{F} = \dfrac{FSR}{\Delta\nu_{1/2}} = \dfrac{\pi\sqrt{R}}{1-R}$ (Etalon Finesse) ①

Sources of cavity (etalon loss)

① Imperfection of reflection at mirrors

② Absorption and scattering loss in the medium between the mirrors.

round-trip reflection

$$R = r_1 r_2 \cdot \underbrace{\exp(-2\alpha_s l)}_{\text{absorption loss}} = \exp(\underbrace{-2\alpha_r l}_{\substack{\text{overall distributive loss} \\ \text{unit: } (cm^{-1})}}) \quad (2)$$

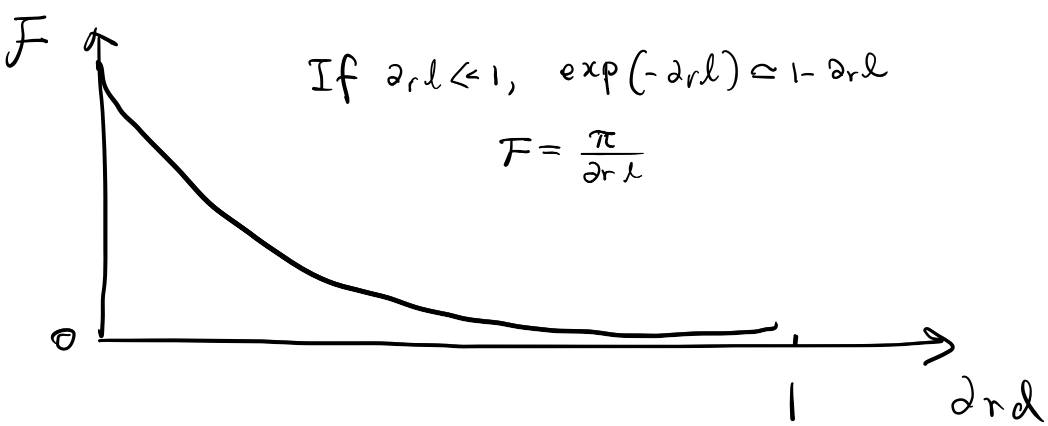
$$\alpha_r = \alpha_s + \frac{1}{2l} \ln \frac{1}{r_1 r_2}$$

$$= \alpha_s + \underbrace{\frac{1}{2l} \ln \frac{1}{r_1}}_{\text{loss of mirror 1}} + \underbrace{\frac{1}{2l} \ln \frac{1}{r_2}}_{\text{loss of mirror 2}}$$

$$= \alpha_s + \alpha_{m1} + \alpha_{m2}$$

Plug in (2) into (1);

$$F = \frac{\pi \exp\left(-\frac{\alpha_r l}{2}\right)}{1 - \exp(-\alpha_r l)}$$



If $\alpha_r l \ll 1$, $\exp(-\alpha_r l) \simeq 1 - \alpha_r l$

$$F = \frac{\pi}{\alpha_r l}$$

Summary:

① Finesse is defined as $\mathcal{F} = \frac{FSR}{\Delta\nu_{1/2}}$

② Finesse is a measure of cavity reflector loss and absorption/scattering loss in the etalon media.

② Photon lifetime.

$$\Delta\nu_{1/2} = \frac{FSR}{F} \simeq \frac{c}{2n\ell} \Big/ \frac{\pi}{\partial_r \ell} = \frac{c\,\partial_r}{2\pi n}$$

$$\simeq \frac{1}{2\pi \tau_p}$$

③ Quality factor:

$$Q = 2\pi \cdot \frac{\text{stored energy}}{\text{energy loss per cycle}}$$

Stored energy E, loss rate: $\frac{c}{n} \cdot \partial_r E$,

loss per cycle: $\frac{c\,\partial_r E}{n\nu}$

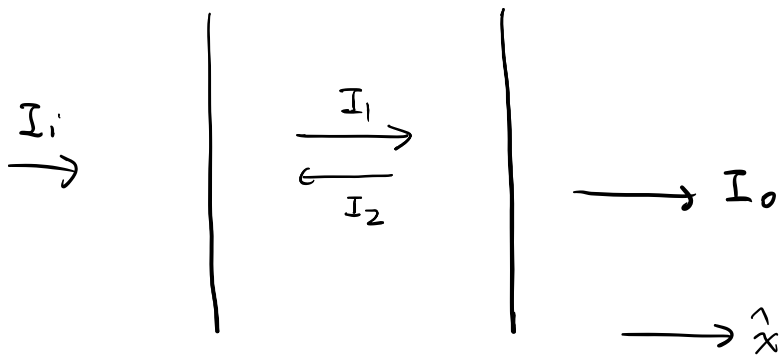
$$\Rightarrow Q = 2\pi \cdot \frac{E}{\frac{c\,\partial_r E}{n\,\nu}} = \frac{2\pi\nu n}{c\,\partial_r} \simeq \frac{\nu}{\nu_{1/2}}$$

It can also be shown that

$$Q \simeq \frac{\nu}{FSR} \cdot \mathcal{F}$$

Usually, $\nu \gg FSR$, $Q \gg \mathcal{F}$

4. Intracavity field



The $\vec{\mathcal{E}}$-field inside cavity can be written as the sum of right traveling wave (I_1) and left-traveling wave (I_2)

Assuming symmetric etalon:

$$I_o = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2\left(\frac{\delta}{2}\right)} I_i$$

$$\begin{cases} I_o = T \cdot I_1 = (1-R) I_1 \\ I_2 = R \cdot I_1 \end{cases}$$

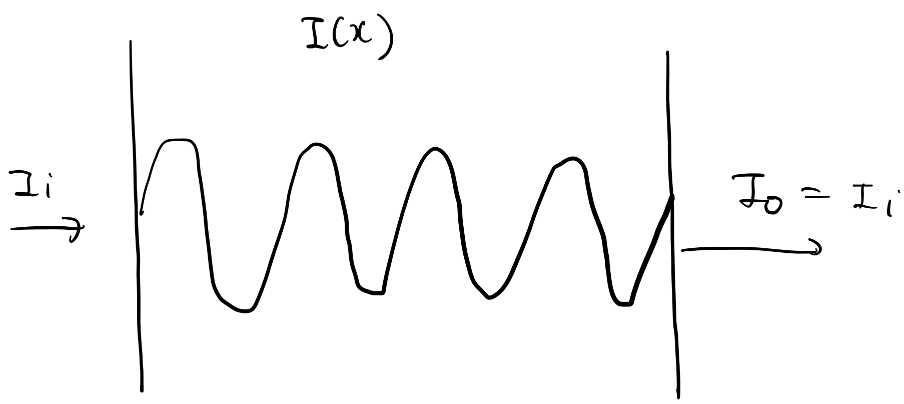
$$\Rightarrow \left.\begin{aligned} I_1 &= \frac{I_o}{1-R} \overset{\text{on resonance}}{=\!=} \frac{1}{1-R} I_i \\ I_2 &= \frac{R}{1-R} I_o =\!= \frac{R}{1-R} I_i \end{aligned}\right\} \text{(1)}$$

When on resonance, the right traveling and left traveling waves form an inteference (standing wave) pattern inside the cavity:

intensity pattern ($I(x)$) inside the cavity

$$I(x) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(2kx + \phi_0) \qquad (2)$$

On resonance (standing wave in cavity)



plug in (1) into (2) and average,

$$\boxed{I_{avg} = \frac{1+R}{1-R} I_i}$$

If $R = 0.99$, when on resonance (standing wave is formed inside cavity), I_{avg} is 199 times higher than I_i

What happens to the transmitted E-field?

The etalon transmission coefficient

$$t_e = \frac{A_t}{A_i} = \frac{Te^{-\frac{i\delta}{2}}}{1-Re^{-i\delta}} \overset{\phi=\frac{\delta}{2}}{=\!=} \frac{Te^{-i\phi}}{1-Re^{-2i\phi}}$$

$$= |t_e| e^{-i\psi}$$

phase shift of transmitted field

$$\psi = \phi + \tan^{-1}\left(\frac{R\sin 2\phi}{1-R\cos 2\phi}\right)$$

the flight time (or group delay) of etalon

$\Delta\phi = \omega\cdot\Delta t - k\cdot\Delta z$

$$\tau_e = \frac{d\psi}{d\omega} = \frac{d\psi}{d\phi}\cdot\frac{d\phi}{d\omega} = \frac{d\psi}{d\phi}\cdot\tau \qquad \Delta t = \frac{d\phi}{d\omega}$$

single trip flight time (group delay)

When on resonance, $\phi = \pi, 2\pi, 3\pi$

$$\frac{d\psi}{d\phi} = \left(\frac{1+R}{1-R}\right) \quad (\text{when } \phi = \pi, 2\pi, 3\pi)$$

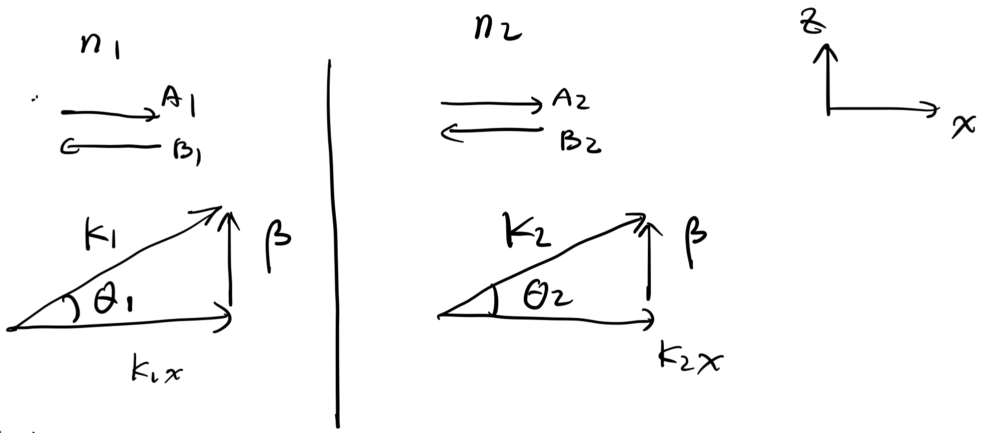
So, $\tau_e = \left(\frac{1+R}{1-R}\right)\tau$,

For $R = 0.99$, the time of fight (group delay) is enhanced by a factor of 199.

Summary:

5. Multicavity elaton and transfer matrix

① At interface



ignore $i\omega t$ terms.

right- left-traveling

$$E = \begin{cases} (A_1 e^{-ik_1 x} + B_1 e^{ik_1 x}) e^{-i\beta z} & \text{medium 1} \\ (A_2 e^{-ik_2 x} + B_2 e^{ik_2 x}) e^{-i\beta z} & \text{medium 2} \end{cases}$$

where
$$k_{1x}^2 + \beta^2 = k_1^2 = \left(n_1 \frac{\omega}{c}\right)^2$$
$$k_{2x}^2 + \beta^2 = k_2^2 = \left(n_2 \frac{\omega}{c}\right)^2$$

$$\begin{cases} B_1 = A_1 \cdot r_{12} + B_2 \cdot t_{21} \\ A_2 = A_1 t_{12} + B_2 r_{21} \end{cases}$$

where r_{12}, r_{21}, t_{12} t_{21} are Fresnel reflection and transmission coefficients

$\Rightarrow$

$$A_1 = \frac{1}{t_{12}} A_2 - \frac{r_{21}}{t_{12}} B_2$$

$$B_1 = \frac{r_{12}}{t_{12}} A_2 + \left(t_{21} - \frac{r_{12} \cdot r_{21}}{t_{12}} \right) B_2$$

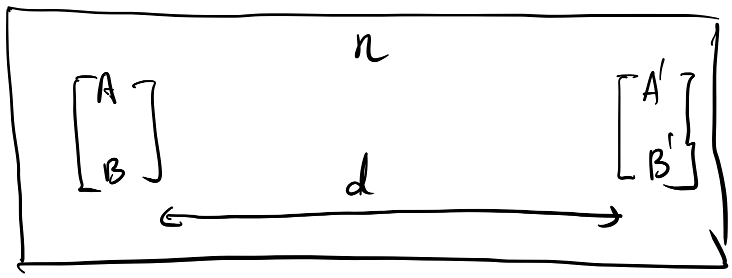
$r_{21} = -r_{12}$,

$t_{12} t_{21} - r_{12} r_{21} = 1$

$$\Rightarrow \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \frac{1}{t_{12}} \begin{bmatrix} 1 & r_{21} \\ r_{12} & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \overset{\text{transfer matrix}}{T_{12}} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

where T_{12} is defined as the transfer matrix between layer 1 and 2. T_{12} is asymmetric matrix.

(2) in homogeneous medium



propagation of plane wave in a homogeneous medium

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \underset{\text{propagation matrix}}{P} \begin{bmatrix} A' \\ B' \end{bmatrix}$$

phase shift is given by

$$\phi = k_x \cdot d = n \cdot \frac{\omega}{c} \cdot \cos\theta \, d.$$

If there are N+1 media,

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix} \equiv M \begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix}$$

where $M = T_{01} \cdot P_1 \cdot T_{12} \cdot P_2 \cdots P_{N-1} \cdot T_{N-1,N} \cdot P_N \cdot T_{N,N+1}$

$M_{21} = M_{12}^*$, $\quad M_{22} = M_{11}^*$ for lossless multilayer.

Determinant of M :

$$|M| = \frac{k_{N+1\,x}}{k_{0x}}$$

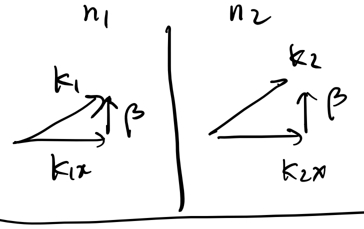
Recall that Fresnel coefficient :

$$r_{12} = \begin{cases} \dfrac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = \dfrac{k_{1x} - k_{2x}}{k_{1x} + k_{2x}} & (\text{s-wave}) \\[2ex] \dfrac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = \dfrac{n_1^2 k_{2x} - n_2^2 k_{1x}}{n_1^2 k_{2x} + n_2^2 k_{1x}} & (\text{p-wave}) \end{cases}$$

$$t_{12} = \begin{cases} \dfrac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = \dfrac{2 k_{1x}}{k_{1x} + k_{2x}} & (\text{s-wave}) \\[2ex] \dfrac{2 n_1 \cos\theta}{n_2 \cos\theta_1 + n_1 \cos\theta_2} = \dfrac{2 n_1 n_2 k_{1x}}{n_1^2 k_{2x} + n_2^2 k_{1x}} & (\text{p-wave}) \end{cases}$$

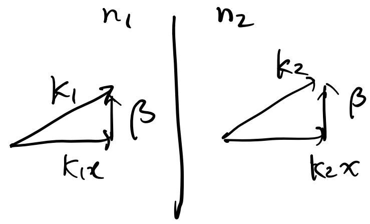
Summary

s- polarization



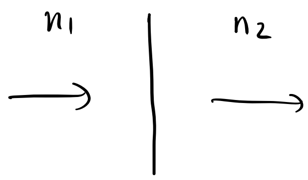
$$T_{12} = \frac{1}{2k_{1x}} \begin{bmatrix} k_{1x}+k_{2x}, & k_{1x}-k_{2x} \\ k_{1x}-k_{2x}, & k_{1x}+k_{2x} \end{bmatrix}$$

p- polarization



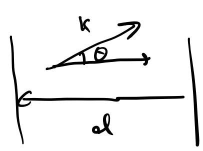
$$T_{12} = \frac{1}{2n_1 n_2 k_{1x}} \begin{bmatrix} n_1^2 k_{2x}+n_2^2 k_{1x}, & n_1^2 k_{2x}-n_2^2 k_{1x} \\ n_1^2 k_{2x}-n_2^2 k_{1x}, & n_1^2 k_{2x}+n_1^2 k_{1x} \end{bmatrix}$$

Normal incidence



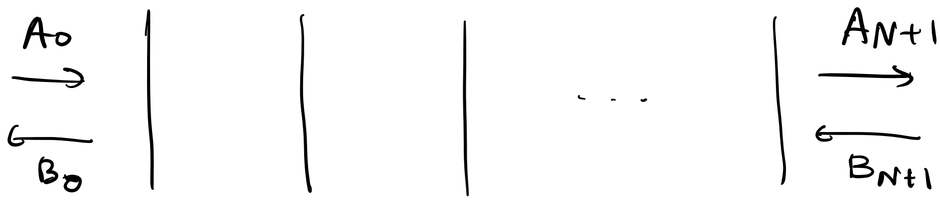
$$T_{12} = \frac{1}{2n_1} \begin{bmatrix} n_1+n_2, & n_1-n_2 \\ n_1-n_2, & n_1+n_2 \end{bmatrix}$$

homogeneous media



$$P = \begin{bmatrix} e^{ik_x d} & 0 \\ 0 & e^{-ik_x d} \end{bmatrix}$$

Consider $N+1$ layers and N interfaces.



$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix} \equiv M \begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix}$$

where $M = T_{01} P_1 \cdot T_{12} \cdot P_2 \cdot T_{23} \cdot P_3 \cdots P_{N-1} T_{N-1\,N} \cdot P_N \cdot T_{N,N+1}$

$M_{21} = M_{12}^*$, $M_{22} = M_{11}^*$ for lossless multilayer

Determinant of M:

$$\det(M) = \frac{k_{N+1\,x}}{k_{0x}}$$

① If light is launched from the left side, $B_{N+1} = 0$

$$A_0 = M_{11} A_{N+1}$$

$$B_0 = M_{21} A_{N+1}$$

total transmission / reflection coefficients:

$$t = \left(\frac{A_{N+1}}{A_0}\right) = \frac{1}{M_{11}}$$

$$r = \left(\frac{B_0}{A_0}\right) = \frac{M_{21}}{M_{11}}$$

② If light is launched from the right side ; $A_0 = 0$

$$t' = \left(\frac{B_0}{B_{N+1}}\right) = \frac{\det(M)}{M_{11}} \qquad ①$$

$$r' = \left(\frac{A_{N+1}}{B_{N+1}}\right) = -\frac{M_{12}}{M_{11}} \qquad ②$$

Power transmittance

From left to right : $T = \frac{k_{N+1}}{k_{0x}} |t|^2$

right to left : $T' = \frac{k_{0x}}{k_{N+1\,x}} |t'|^2$

plug in ①, ② , we have

$$\boxed{T = T'}$$