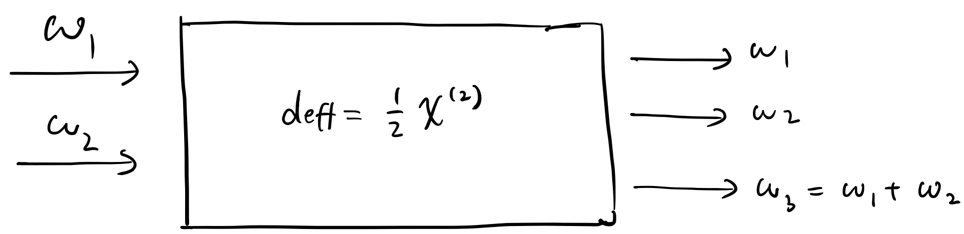
Lecture 5: $\chi^{(2)}$ nonlinear optical processes I (SFG, SHG)

Learning objectives:

1. SFG under undepleted pump approximation
2. SHG (Generic case)
3. SHG in QPM nanophotonic waveguides

1. Sum-frequency generation (SFG)

Recap.



CAEs:

$$
\begin{cases}
\dfrac{dA_3}{dz} = \dfrac{2i\, d_{eff}\, \omega_3^2}{k_3 c^2} A_1 A_2 e^{i\Delta k z} \\[2ex]
\dfrac{dA_1}{dz} = \dfrac{2i\, d_{eff}\, \omega_1^2}{k_1 c^2} A_3 A_2^* e^{-i\Delta k z} \\[2ex]
\dfrac{dA_2}{dz} = \dfrac{2i\, d_{eff}\, \omega_2^2}{k_2 c^2} A_3 A_1^* e^{-i\Delta k z}
\end{cases}
$$

How to solve $A_3(z)$, $A_2(z)$, $A_1(z)$?

Use Jacobi elliptic function! (Armstrong et al. 1962)

<u>Difficulty</u>: each eq. has multiple variables.

Undepleted pump approximation:

A_2 is very strong, has no change along z ($A_2 = \text{Const.}$)

eg: upconversion for long-wavelength light detection

CAEs:

$$\begin{cases} \dfrac{dA_3}{dz} = \underbrace{\boxed{\dfrac{2i d_{eff}\,\omega_3^2}{k_3 c^2}}}_{k_3} A_2 A_1 e^{i\Delta k z} \\ \dfrac{dA_1}{dz} = \underbrace{\boxed{\dfrac{2i d_{eff}\,\omega_1^2}{k_1 c^2}}}_{k_1} A_2^* A_3 e^{-i\Delta k z} \\ \dfrac{dA_2}{dz} = \dfrac{2i d_{eff}\cdot\omega_2^2}{k_2 c^2} A_3 \cdot A_1^* e^{-i\Delta k z} \end{cases}$$

① $\Delta k = 0$ (perfect phase matching)

CAEs:

$$\begin{cases} \dfrac{dA_1}{dz} = k_1 A_3 & ① \\ \dfrac{dA_3}{dz} = k_3 A_1 & ② \end{cases}$$

$\frac{d}{dz}$①, we get $\dfrac{d^2A_1}{dz} = k_1 \dfrac{dA_3}{dz}$

$$\Rightarrow \frac{d^2A_1}{dz} = k_1 k_3 \cdot A_1 = -k^2 \cdot A_1 \qquad ③$$

where $k^2 \equiv -k_1 k_3 = \dfrac{4\omega_1^2\omega_3^2 d_{eff}^2 |A_2|^2}{k_1 k_3 c^4}$

Solution of ③,

$$A_1(z) = B\cos kz + C\sin kz.$$

and

$$A_3(z) = \left(\frac{dA_1}{dz}\right)\Big/K_1 = -\frac{Bk}{K_1}\sin kz + \frac{Ck}{K_1}\cos kz$$

Boundary conditions:

at $z=0$, $A_3 = 0 \Rightarrow C = 0$

$A_1 = A_1(0) \Rightarrow B = A_1(0)$

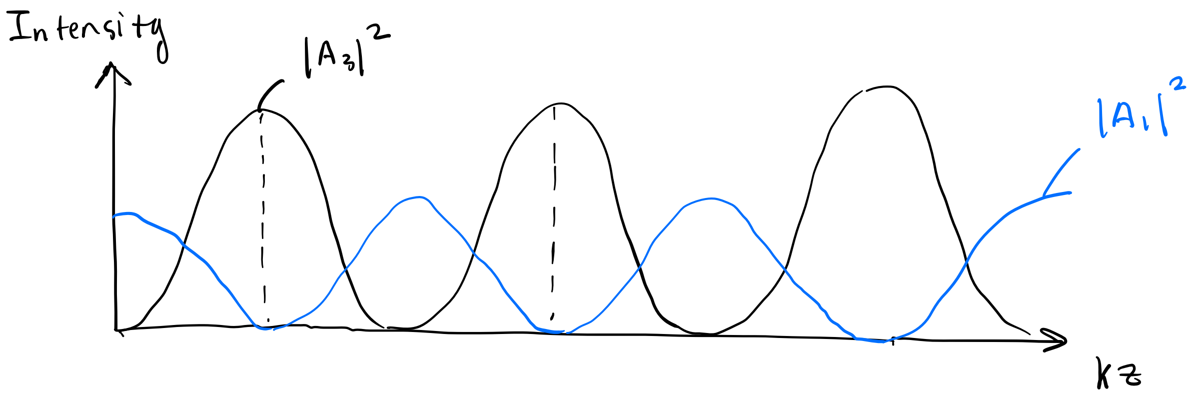
$$\Rightarrow \begin{cases} A_1(z) = A_1(0)\cos kz \\ A_3(z) = -A_1(0)\dfrac{k}{K_1}\sin kz \end{cases}$$

$\underbrace{\dfrac{k}{K_1}}_{=?}$

$$\frac{k}{K_1} = \frac{2\omega_1\omega_3 d_{eff}|A_2|}{(k_1k_3)^{1/2}\cdot c^2}\cdot\frac{k_1c^2}{2i\omega_1^2 d_{eff}\cdot A_2^*} = -i\left(\frac{n_1\omega_3}{n_3\omega_1}\right)^{1/2}\frac{|A_2|}{A_2^*}$$

and $$\frac{|A_2|}{A_2^*} = \frac{A_2}{A_2}\cdot\frac{|A_2|}{A_2^*} = \frac{A_2|A_2|}{|A_2|^2} = \frac{A_2}{|A_2|} = e^{i\phi_2}$$

So,
$$\begin{cases} A_3(z) = i\left(\dfrac{n_1\omega_3}{n_3\omega_1}\right)^{1/2} \cdot A_1(0)\sin kz \, e^{i\phi_2} \\ A_1(z) = A_1(0)\cdot\cos kz. \end{cases}$$



② $\Delta k \neq 0$ (phase mismatched)

$$\begin{cases} \dfrac{dA_1}{dz} = k_1 A_3 e^{-i\Delta kz} & ① \\ \dfrac{dA_3}{dz} = k_3 A_1 e^{i\Delta kz} & ② \end{cases}$$

Trial Solutions:

$$\begin{cases} A_1(z) = (F e^{\overset{\text{spatial variation}}{igz}} + G e^{-igz}) e^{-i\Delta kz/2} \\ A_3(z) = (C e^{igz} + D e^{-igz}) e^{i\frac{\Delta kz}{2}} \end{cases}$$

plug in ①. ②,

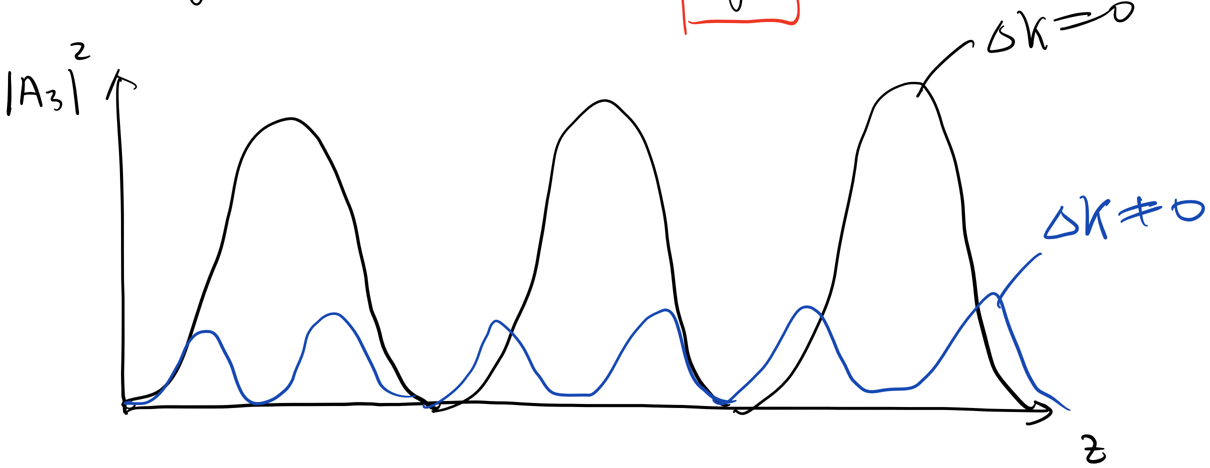
$$A_1(z) = \left[A_1(0)\cos gz + \left(\frac{k_1}{g}A_3(0) + \frac{i\Delta k}{2g}A_1(0)\right)\sin gz\right] e^{-\frac{1}{2}i\Delta k z}$$

$$A_3(z) = \left[A_3(0)\cos gz + \left(-\frac{i\Delta k}{2g}A_3(0) + \frac{k_3}{g}A_1(0)\right)\sin gz\right] e^{\frac{1}{2}i\Delta k z}$$

Let $A_3(0) = 0$

$$A_3(z) = \frac{k_3}{g} \cdot A_1(0) \sin gz \, e^{\frac{1}{2}i\Delta k z}.$$

Intensity $\sim |A_3(z)|^2 = |A_1(0)|^2 \boxed{\frac{|k_3|^2}{g^2}} \sin^2 \boxed{gz}$, $g = \sqrt{k^2 + \frac{1}{4}\Delta k^2}$



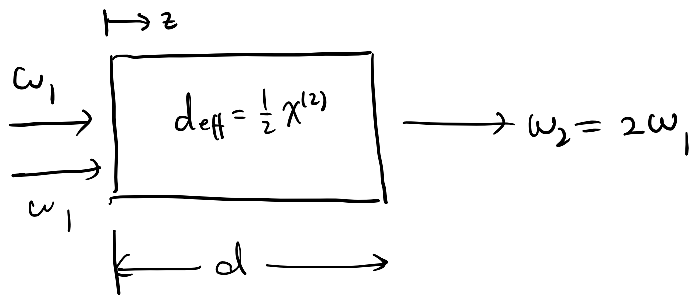
Comments:

1. When $\Delta k \neq 0$, there are more "oscillations"

2. As $\Delta k \uparrow$, the maximum intensity is decreased by the factor $\dfrac{|k_3|^2}{k^2 + \frac{1}{4}\Delta k^2}$

2. Second harmonic generation

SHG can be regarded as a specical case of SFG



$\Delta k = k_1 + k_1 - k_2$
$= 2k_1 - k_2$

CAEs:

$$\begin{cases} \frac{dA_1}{dz} = \frac{2i\omega_1^2 d_{eff}}{k_1 c^2} A_2 A_1^* e^{-i\Delta k z} & ① \\ \frac{dA_2}{dz} = \frac{i\omega_2^2 d_{eff}}{k_2 c^2} A_1^2 e^{i\Delta k z} & ② \end{cases}$$

① Under undepleted pump approx. $A_1 = $ Const

A_2 can be calculated by direct integration of ②

If $\Delta k = 0$. $A_2(z) \sim z$, $I_2 \sim z^2$ (only valid at low pump)

② Generic case, ① and ② should be solved together.

spatially varying term

$$A_1 = \left(\frac{I}{2n_1 \epsilon_0 c}\right)^{1/2} u_1(z) e^{i\phi_1}$$

$$A_2 = \left(\frac{I}{2n_2 \epsilon_0 c}\right)^{1/2} u_2(z) e^{i\phi_2}$$

Where total intensity: $I = I_1 + I_2 = 2n_1 \epsilon_0 c |A_1|^2 + 2n_2 \epsilon_0 c |A_2|^2$

which requires: $u_1^2(z) + u_2^2(z) = 1$

introduce normalized distance:

$$\zeta = \frac{z}{l} = \frac{z}{\left(\frac{2n_1^2 n_2}{\epsilon_0 c I}\right)^{1/2} \cdot \frac{c}{2\omega_1 d_{eff}}}$$

"l" characteristic distance, energy exchange.

relative phase of interacting fields:

$$\theta = 2\phi_1 - \phi_2 + \Delta k z.$$

let $\Delta s = \Delta k l$ be the normalized phase mismatch

Then, the CAEs can be rewritten as:

$$\begin{cases} \frac{du_1}{d\zeta} = u_1 u_2 \sin\theta & (1) \\ \frac{du_2}{d\zeta} = -u_1^2 \sin\theta & (2) \\ \frac{d\theta}{d\zeta} = \Delta s + \frac{\cos\theta}{\sin\theta} \frac{d}{d\zeta}(\ln u_1^2 u_2) & (3) \end{cases}$$

When perfect phase matching ($\Delta k = 0$, $\Delta s = 0$) (3) can be written as

$$\frac{d}{d\zeta} \ln(\underbrace{u_1^2 u_2 \cos\theta}_{P}) = 0$$

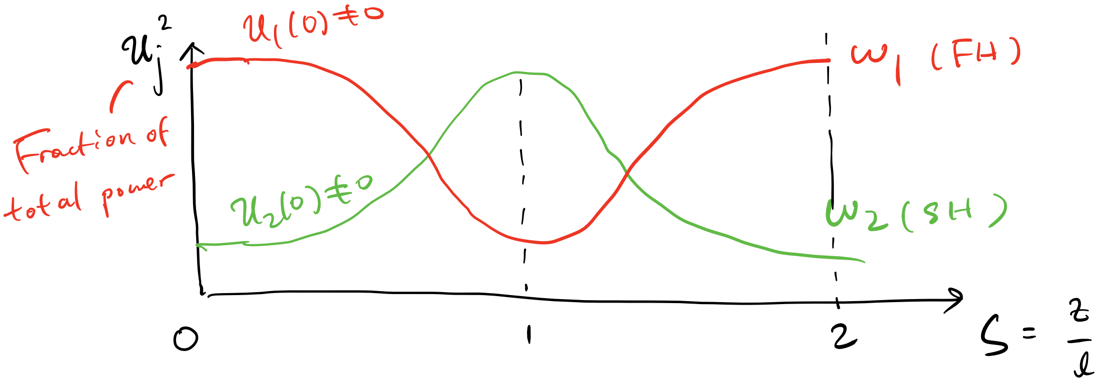
$\Rightarrow$ $\ln P$ is a constant, doesn't depend on z

can be calculated by $u_1(0)$ $u_2(0)$ and θ

①, ② can be written as

$$\boxed{\frac{du_2^2}{d\zeta} = \pm 2\left[(1-u_2^2)\,u_2^2 - \Gamma^2\right]^{1/2}}$$

Solution: Jacobi elliptic functions



Comments:

① Generic case, $(\Gamma \neq 0)$, FH and SH exchange energy periodically.

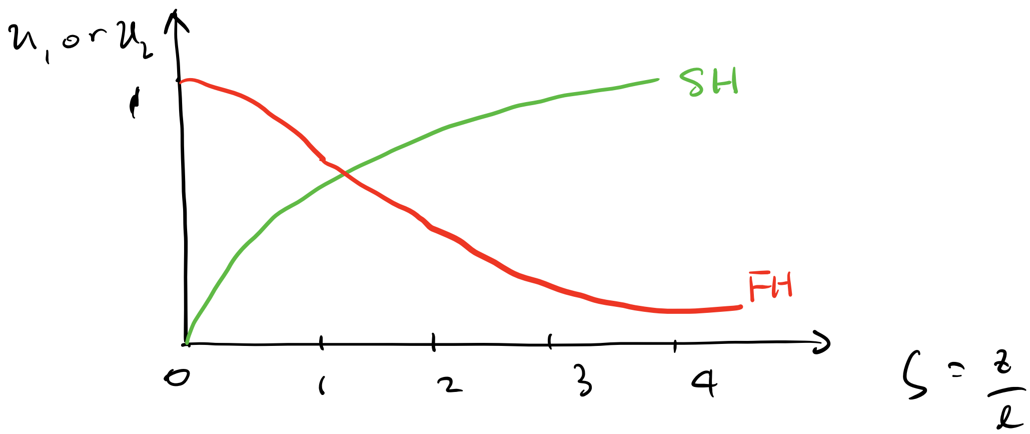
② when $\Gamma = 0$, (no SH input, or initially phased, $\cos\theta = 0$)

$\Gamma = u_1^2 u_2 \cos\theta$ is conserved during propagation

So $\cos\theta = 0, \Rightarrow \sin\theta = -1$

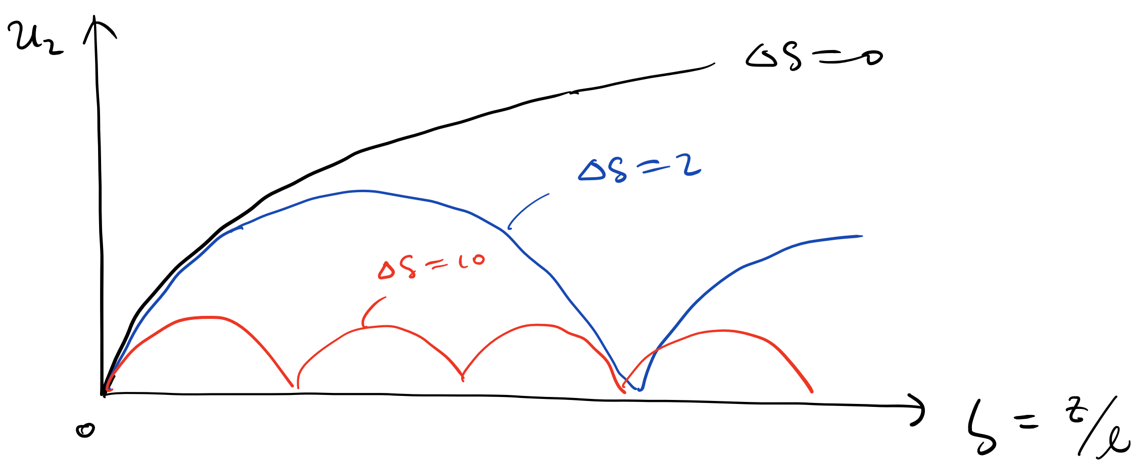
CAEs becomes
$$\begin{cases} \dfrac{du_1}{d\zeta} = -u_1 u_2 \\ \dfrac{du_2}{d\zeta} = u_1^2 \end{cases} \Rightarrow \begin{cases} u_1(\zeta) = \operatorname{sech}\zeta \\ u_2(\zeta) = \tanh\zeta \end{cases}$$

Solution:

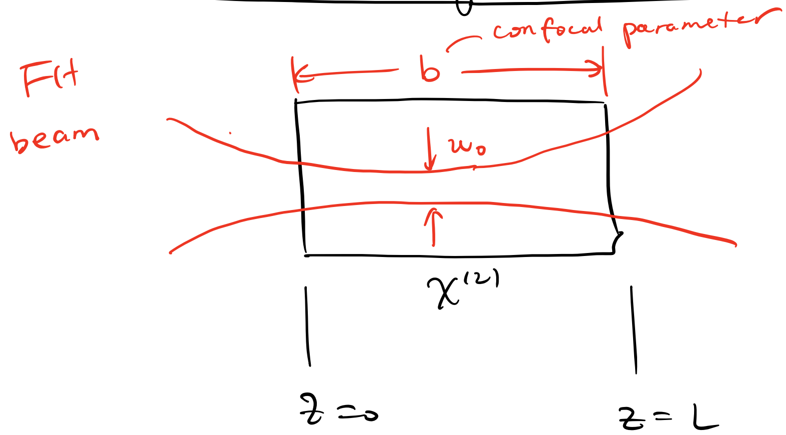


No backconversion, when $\zeta = \infty$, all FH is converted into SH, $l=$

③ When $\Delta k \neq 0$ ($\Delta s \neq 0$)



SH conversion efficiency estimation.



Assumption: incident FH beam is focused to the center of the crystal, with power P. Depth of focal region = L ($b=L$), perfect phase-matching

Intensity of input FH beam:

$$I_1 = \frac{P}{\pi w_0^2} = 2n_1 \epsilon_0 c \cdot A_1^2$$

$$b = \frac{2\pi w_0^2}{\lambda_1/n_1} = L \qquad \text{(P118. Boyd)}$$

$$\Rightarrow \quad A_1 = \sqrt{\frac{P}{\epsilon_0 c \cdot \lambda_1 L}}.$$

$$\zeta = \frac{L}{l} = \frac{L}{\dfrac{\sqrt{n_1 n_2}\cdot c}{2\omega_1 d_{eff} \cdot |A_1(0)|}} = \sqrt{\frac{16\pi^2 d_{eff}^2 \cdot LP}{\epsilon_0 c\, n_1 n_2 \lambda_1^3}}$$

Typical values:

$d_{eff} = 4\times10^{-12}$ m/V, $L = 1$ cm, $P = 1$ W, $\lambda = 0.5\,\mu$m,

$n = 2$

$\Rightarrow \zeta = 0.14$.

Conversion efficiency: $\eta = \dfrac{P_{SH}}{P_{FH}} = \dfrac{u_2^2(\zeta)}{u_1^2(0)} \sim 2\%$.

How to improve efficiency?

SHG in QPM nanophotonic waveguide

Applications of SHG?

1. Nonlinear optical microscopy (two-photon)
2. Material characterization