Lecture 4: Phase-matching and Quasi-phase-matching

Learning objectives:

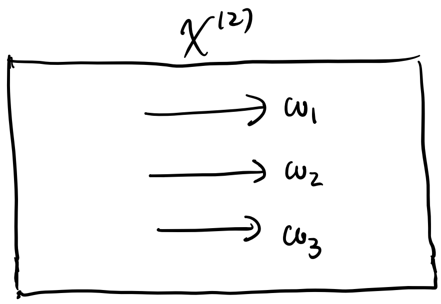
1. The Manley-Rowe relation

2. Difficulties in phase-matching

3. Techiques for phase matching

   - Birefrigence crystal
   - Quasi-phase matching

1. The Manley-Rowe relations.



Three-wave interation in $\chi^{(2)}$ media

$$
\begin{cases}
\dfrac{dA_3}{dz} = \dfrac{2i\, d_{eff}\, \omega_3^2}{k_3 c^2} A_1 A_2 e^{i\Delta k z} & (1) \\[2ex]
\dfrac{dA_1}{dz} = \dfrac{2i\, d_{eff}\, \omega_1^2}{k_1 c^2} A_3 \cdot A_2^* e^{-i\Delta k z} & (2) \\[2ex]
\dfrac{dA_2}{dz} = \dfrac{2i\, d_{eff} \cdot \omega_2^2}{k_2 c^2} A_3 \cdot A_1^* e^{-i\Delta k z} & (3)
\end{cases}
$$

(evloution of field amplitudes for SFG)

Intensity of each wave:

$$I_i = 2 n_i \epsilon_0 c \cdot A_i A_i^*$$

$$\frac{dI_i}{dz} = 2 n_i \epsilon_0 c \left(A_i^* \frac{dA_i}{dz} + A_i \frac{dA_i^*}{dz} \right)$$

For I_1 :

plug in (2)

$$\frac{dI_1}{dz} = 2 n_1 \epsilon_0 c \cdot \left(A_1^* \cdot \frac{dA_1}{dz} + A_1 \frac{dA_1^*}{dz} \right)$$

$$\Rightarrow \frac{dI_1}{dz} = 2 n_i \epsilon_0 c \cdot \frac{2 d_{eff}\, \omega^2}{k_1 c^2} \left(i A_1^* A_3 \cdot A_2^* \cdot e^{-i\Delta k z} + c.c. \right)$$

$$= 4 \epsilon_0 d_{eff}\, \omega_1 \left(i A_3 A_1^* A_2^* e^{-i\Delta k z} + c.c. \right)$$

$$= -8\epsilon_0 d_{eff}\,\omega_1\,\mathrm{Im}\left(A_3\cdot A_1^*\cdot A_2^*\cdot e^{-i\Delta kz}\right)$$

Similarly,

$$\frac{dI_2}{dz} = -8\epsilon_0 d_{eff}\,\omega_2\,\mathrm{Im}\left(A_3\cdot A_1^*\cdot A_2^* e^{-i\Delta kz}\right)$$

$$\frac{dI_3}{dz} = -8\epsilon_0 d_{eff}\,\omega_3\cdot\mathrm{Im}\left(A_3^* A_1 A_2 e^{i\Delta kz}\right)$$

$$= 8\epsilon_0 d_{eff}\,\omega_3\cdot\mathrm{Im}\left(A_3\cdot A_1^* A_2^* e^{-i\Delta kz}\right)$$

Comments:

1. For SFG process, $\frac{dI_1}{dz}$ and $\frac{dI_2}{dz}$ are of the same sign.
2. Δk govers the direction of energy flow.
3. Total power flow (energy) is conserved.

Proof:

$$I = I_1 + I_2 + I_3$$

$$\frac{dI}{dz} = \frac{dI_1}{dz} + \frac{dI_2}{dz} + \frac{dI_3}{dz} =$$

$$= -8\epsilon_0 d_{eff}\left(\omega_1+\omega_2-\omega_3\right)\mathrm{Im}\left(A_3 A_1^* A_2^* e^{i\Delta kz}\right) = 0$$

(I is a constant along the crystal)

④ $\frac{d}{dz}\left(\frac{I_1}{\omega_1}\right) = \frac{d}{dz}\left(\frac{I_2}{\omega_2}\right) = -\frac{d}{dz}\left(\frac{I_3}{\omega_3}\right)$ (Manley-Rowe relations)

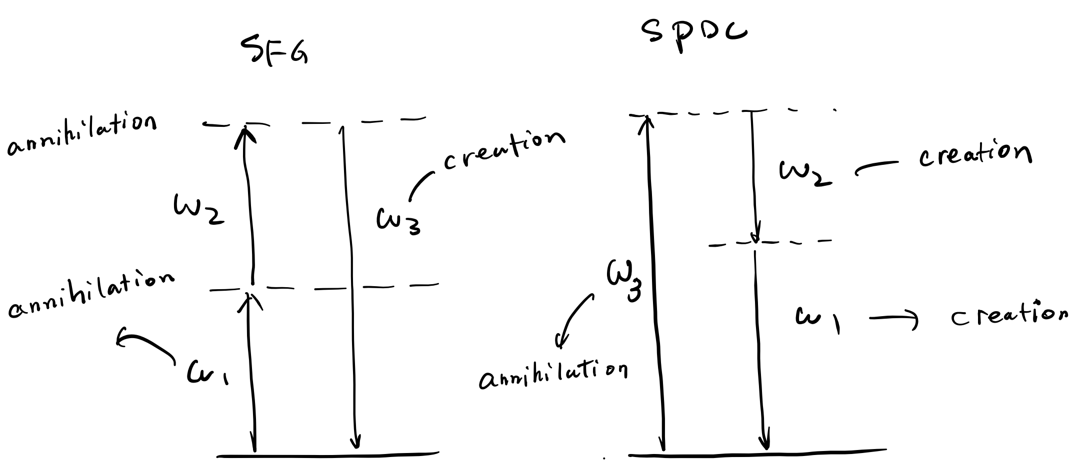
Physical meaning of Manley-Rowe relation?

energy of a photon : $\hbar\omega_i$

$$\text{Intensity} \sim \frac{\text{Power}}{A} = \frac{\text{Energy}/t}{A} = \frac{N \cdot \hbar\omega_i}{t \cdot A}$$

A → area. N → # of photons

$\Rightarrow \frac{I_i}{\omega_i}$ denotes # of photons per unit area, per unit time.



SFG:

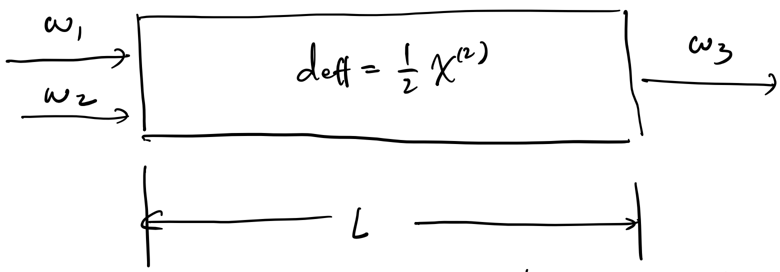
The rate at which photons at ω_3 are created

= the rate at which photons at ω_1 are destroyed

= .. .. ω_2 .. ..

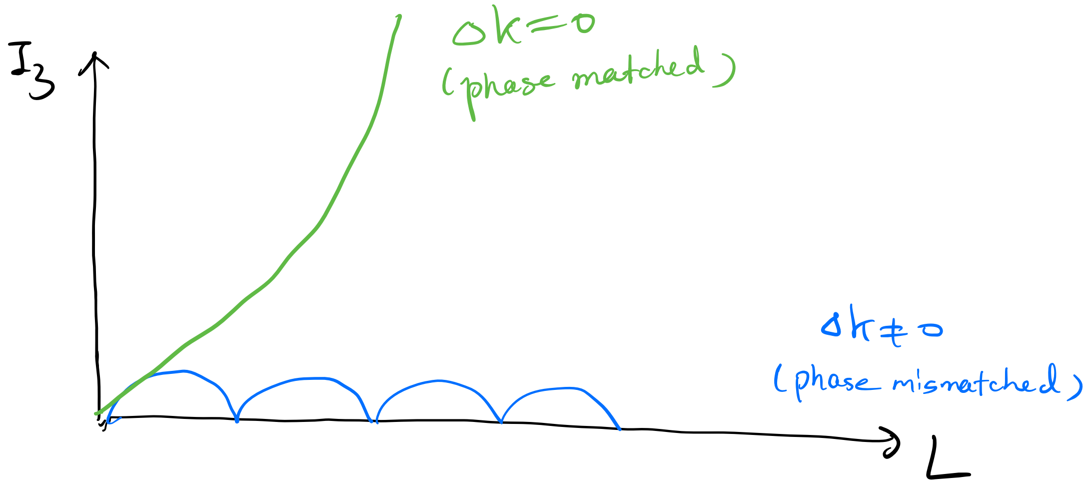
2. Difficulty in phase-matching

For SFG



$\Delta k = k_1 + k_2 - k_3$

$$I_3 \propto \mathrm{sinc}^2\left(\frac{\Delta k L}{2}\right)$$



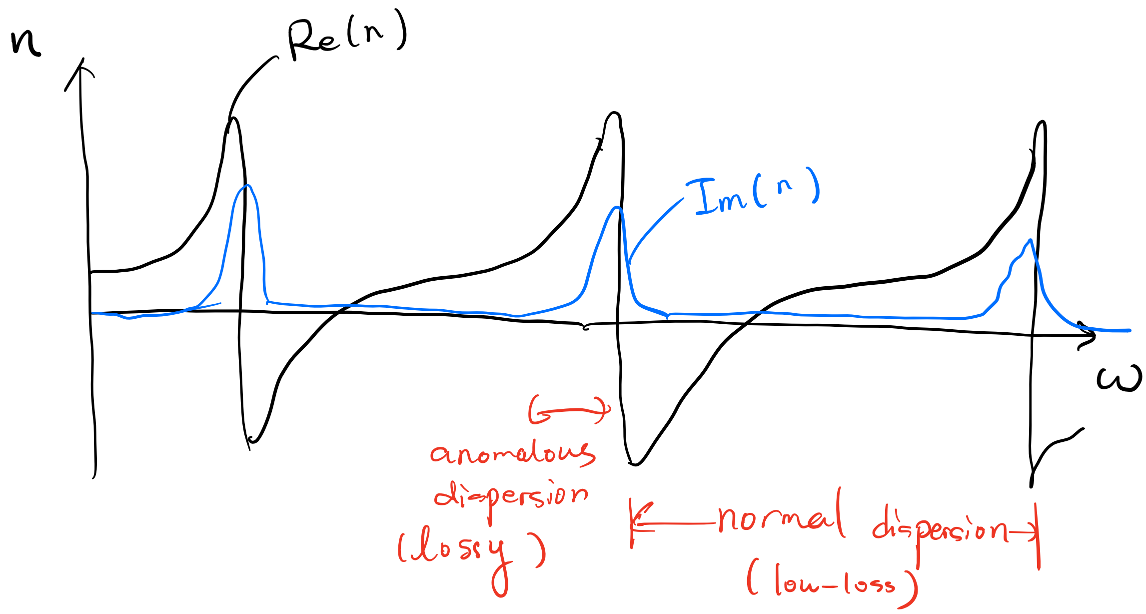
Phase matching condition:

$$\Delta k = k_1 + k_2 - k_3 = 0 \qquad k_i = n_i \cdot \frac{\omega_i}{c}$$

$$\Rightarrow n_1 \frac{\omega_1}{c} + n_2 \frac{\omega_2}{c} = n_3 \frac{\omega_3}{c} \qquad \text{(momentum conservation)}$$

where $\omega_3 = \omega_1 + \omega_2$ (energy conservation)

Dispersion of materials.



Comments:

① For optical materials, in normal dispersion regime $n \uparrow$ as $\omega \uparrow$, (harmonic osillator)

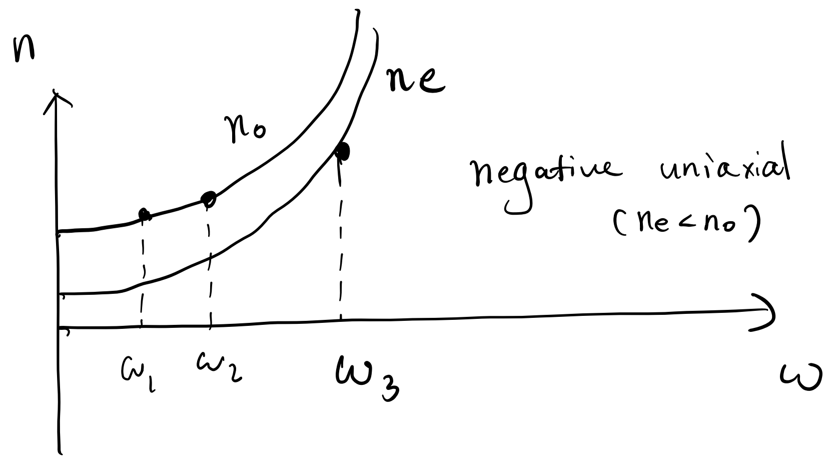
② Since $\omega_3 > \omega_1$, $\omega_3 > \omega_2$. $n_3 > n_1$, $n_3 > n_2$ realizing $n_1 \frac{\omega_1}{c} + n_2 \frac{\omega_2}{c} = n_3 \frac{\omega_3}{c}$ is impossible! as

$$n_1 \frac{\omega_1}{c} + n_2 \frac{\omega_2}{c} < n_3 \frac{\omega}{c}.$$

3. Birefringence crystal. (non-cubic)

Birefringence: dependence of n on the direction of polarization of optical radiation

e.g.



negative uniaxial crystal
($n_e < n_o$)

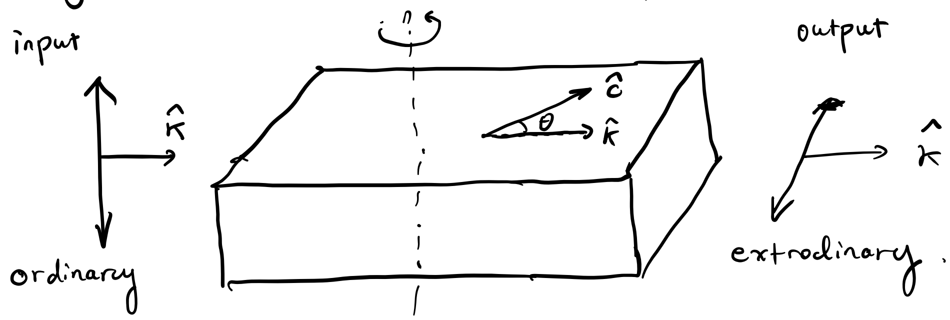
	Positive uniaxial ($n_e > n_o$)	negative uniaxial ($n_e < n_o$)
Type-I	$n_3^o \omega_3 = n_1^e \omega_1 + n_2^e \omega_2$	$n_3^e \omega_3 = n_1^o \omega_1 + n_2^o \omega_2$
Type-II	$n_3^o \omega_3 = n_1^o \omega_1 + n_2^e \omega_2$	$n_3^e \omega_3 = n_1^e \omega_1 + n_2^o \omega_2$

Note: type-I is easier to achieve since ω_1 and ω_2 can share the same polarization

How to realize phase matching in experiments?

Angle tuning:

e.g. Negative uniaxial crystal. ($n_e < n_o$)

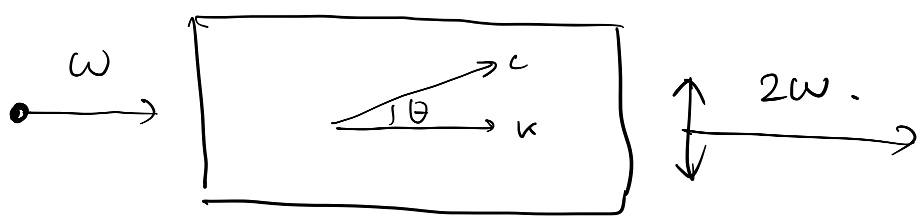


n_e depends on θ

$$\frac{1}{n_e(\theta)^2} = \frac{\sin^2\theta}{\bar{n}_e^2} + \frac{\cos^2\theta}{n_o^2} \qquad (1)$$

principle value of n_e.
$n_e(90^\circ) = \bar{n}_e$

Example: SHG process



phase matching requires:

$$n_e(2\omega, \theta) = n_o(\omega) \qquad (2)$$

plug ② into ①, we have

$$\frac{\sin^2\theta}{\bar{n}_e(2\omega)^2} + \frac{\cos^2\theta}{n_o(2\omega)^2} = \frac{1}{n_o(\omega)^2}$$

$$\Rightarrow \quad \theta = \sin^{-1}\sqrt{\frac{\frac{1}{n_o(\omega)^2} - \frac{1}{n_o(2\omega)^2}}{\frac{1}{\bar{n}_e(2\omega)^2} - \frac{1}{n_o(2\omega)^2}}}$$

To achieve the phase matching, crystal should be oriented in a specific angle !!

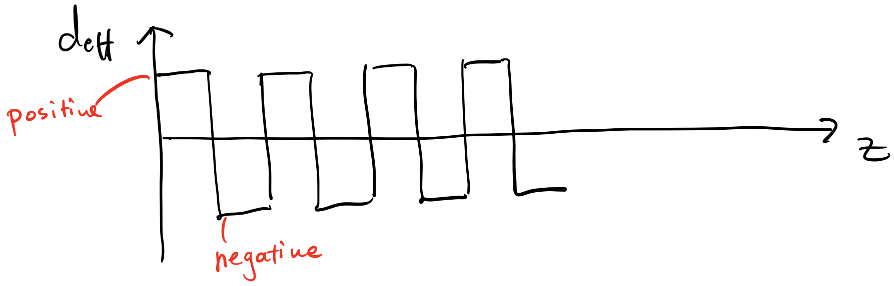
4. Quasi-phase matching

For SFG: $\frac{dA_3}{dz} = C \cdot d_{eff} \cdot e^{i\Delta kz}$. ← we can't do too much with this term

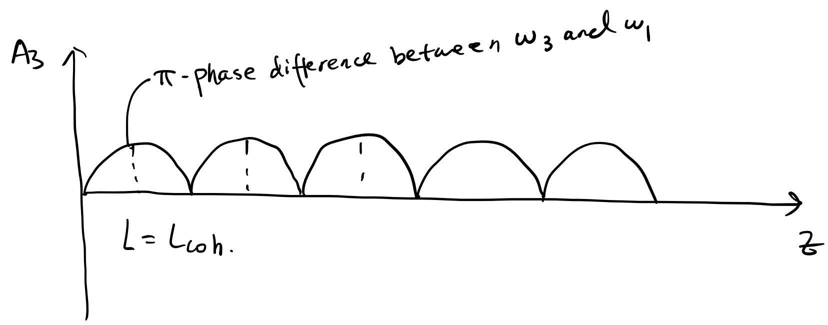
How about this ?

idea case: $d_{eff}(z) = d_{eff} \cdot e^{-i\Delta kz}$ (impossible)

practical case $d_{eff}(z) = d_{eff} \cdot \text{sign}\left[\cos\left(\frac{2\pi z}{\Lambda}\right)\right]$ (QPM)



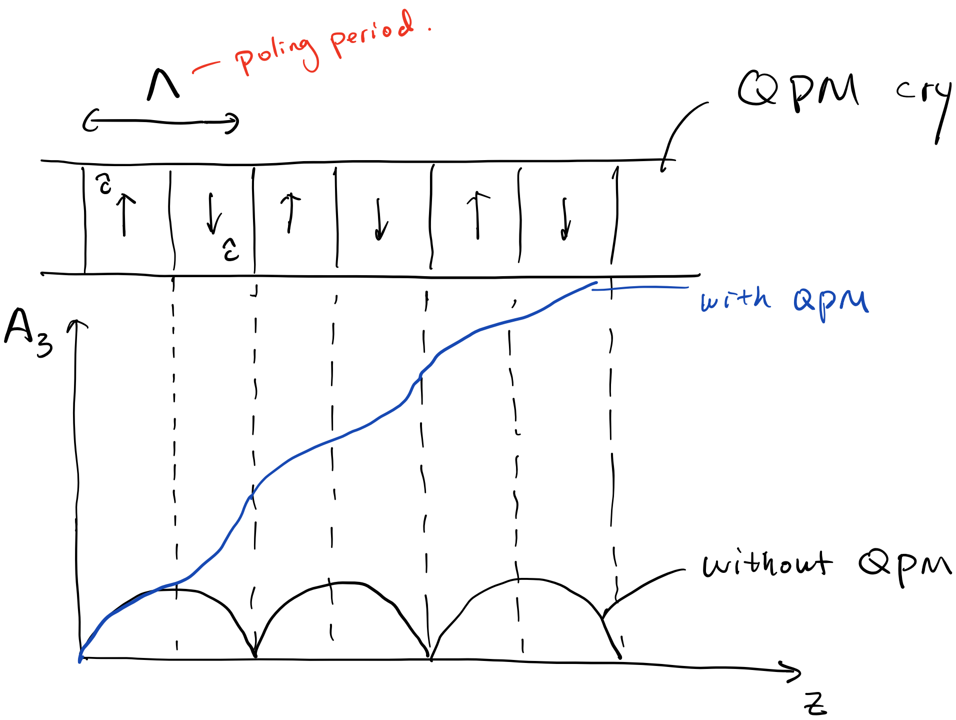
Physical meaning:



At $L = L_{coh}$, there's a π phase difference between ω_3 and ω_1

By introducing QPM, , we flip the sign of d_{eff} at L_{coh}.

then "π phase" is regarded "0-phase" for ω_3 and ω_1 at the next cycle.



$\Lambda = 2 \cdot L_{coh}$.

Mathematics:

For QPM crystals:

$$d(z) = d_{eff}\,\mathrm{sign}\left[\cos\left(\frac{2\pi z}{\Lambda}\right)\right]$$

$$= d_{eff} \cdot \sum_{m=-\infty}^{\infty} G_m \exp(i k_m z)$$

$G_m = \frac{2}{m\pi}\sin\left(\frac{m\pi}{2}\right)$

$k_m = m\frac{2\pi}{\Lambda}$

plug $d(z)$ into coupled amplitude equations, we get:

$$\begin{cases} \dfrac{dA_3}{dz} = \dfrac{2i\omega_3 d_Q}{n_3 c} A_1 A_2 e^{i\Delta k_Q z} \\[2ex] \dfrac{dA_1}{dz} = \dfrac{2i\omega_1 d_Q}{n_1 c} A_3 \cdot A_2^* e^{-i(\Delta k_Q - 2k_m) z} \\[2ex] \dfrac{dA_2}{dz} = \dfrac{2i\omega_2 d_Q}{n_2 c} A_3 A_1^* e^{-i(\Delta k_Q - 2k_m) z} \end{cases}$$

where $d_Q = d_{eff} \cdot G_m = d_{eff} \cdot \frac{2}{m\pi}\sin\left(\frac{m\pi}{2}\right)$

$$\Delta k_Q = k_1 + k_2 - k_3 + \underline{k_m} \leftarrow m \cdot \frac{2\pi}{\Lambda}$$

<u>Comments</u>:

① With QPM, the couple amplitude equations involve modified nonlinear coupling <u>d_{eff}</u>

and phase mismatch Δk.

② when $m=1$. (first order QPM), d_Q is maximum (most efficient)

For 1^{st} order QPM.

$$\Delta k_Q = k_1 + k_2 - k_3 + \frac{2\pi}{\Lambda} = 0$$

$$\Rightarrow \boxed{\Lambda = \frac{2\pi}{k_3 - (k_1 + k_2)}}$$

Recipe for QPM:

① Calculate n_1. n_2. n_3. at ω_1 ω_2. ω_3

② Determine Λ by $\Lambda = \dfrac{2\pi}{n_1 \frac{\omega}{c} - n_2 \frac{\omega_2}{c} - n_3 \frac{\omega_3}{c}}$

③ Fabricate the QPM crystal by

- periodic poling (Feroelectrics, eg. $LiNbO_3$.)
- Orientation patterning (Semiconductors, GaAs, GaP)