Lecture 4. Chromatic dispersion

Learning Objectives:

1. Material dispersion and its origin
2. Complex refractive index/permittivity
3. Optical absorption
4. Kramer-kronig relation and Sellmeir eq.
5. Group velocity and group velocity dispersion (GVD)
6. Optical pulse spreading and frequency chirp

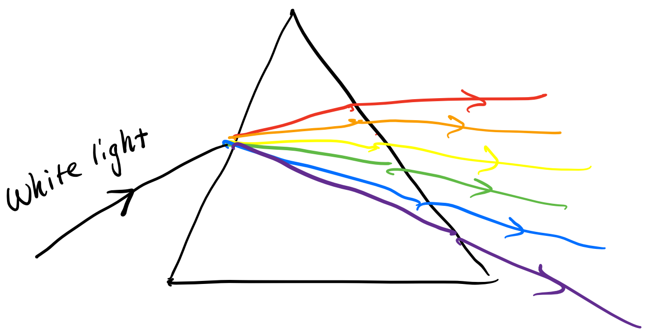
1. Chromatic dispersion and its origin

Propagation of E&M in materials depend on ϵ, and σ, and ϵ, σ are functions of ω, i.e. $\epsilon(\omega)$, $\sigma(\omega)$. In vaccam, there is no dispersion

Examples: 1 Rainbow

2. Dispersion of light in prism

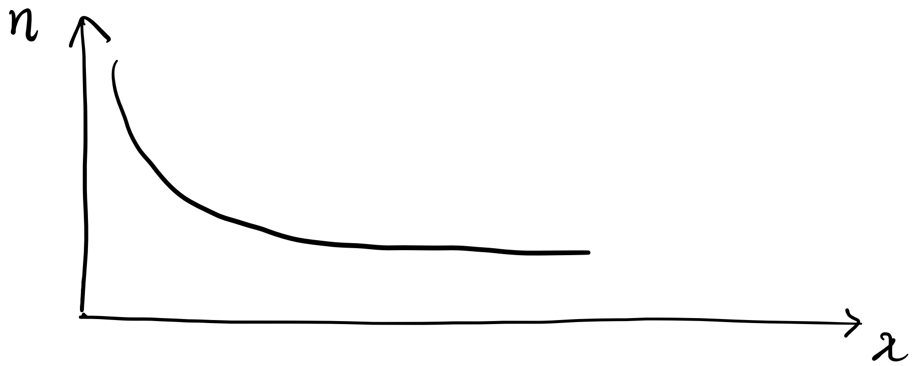
Prism



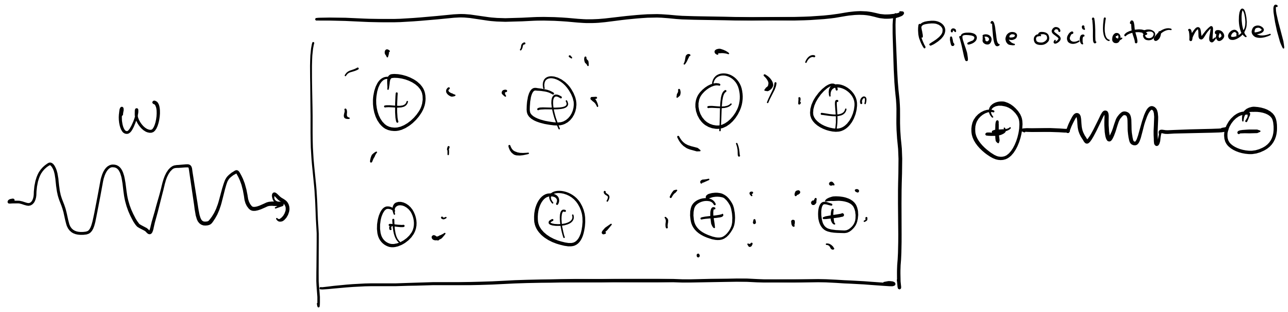
$n = \sqrt{\epsilon_r}$

What does this indicate?

$$n = n(\lambda) \qquad \epsilon = \epsilon(\lambda)$$



Origin of dispersion: dipole oscillation



Assume: nuclear mass >> electron mass, so we ignore the motion of nuclear.

$$F_{binding} = -k_{spring}x = -m\omega_0^2 x$$

mass   disp.

$$F_{damping} = -m\gamma\frac{dx}{dt},$$

With E&M wave of frequency ω, polarised in x direction,

$$F_{driving} = qE = qE_0\cos(\omega t)$$

Newton's second Law:

$$m\frac{d^2x}{dt^2} = F_{binding} + F_{damping} + F_{driving}$$

$$\Rightarrow \quad m\frac{d^2x}{dt} + mr\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos(\omega t)$$

or the real part of the equation

$$\frac{d^2\hat{\tilde{x}}}{dt} + r\frac{d\tilde{x}}{dt} + \omega_0^2 x^2 = \frac{q}{m}E_0 e^{-i\omega t}$$

Assuming $\tilde{x}(t) = \hat{x}_0 e^{i\omega t}$, plug in

$$\tilde{x}_0 = \frac{qE_0}{m(\omega_0^2 - \omega^2 - ir\omega)}$$

Complexed dipole moment induced by $\vec{E}$

$$\hat{\tilde{p}}(t) = q\hat{\tilde{x}}(t) = \frac{q^2}{m(\omega_0^2 - \omega^2 - ir\omega)}E_0 e^{-i\omega t}$$

Comment:

① $\tilde{p}(t)$ is complex, meaning induced polarization might not be in phase with $E(t)$

② phase lag angle: $\tan^{-1}\left(\frac{r\omega}{\omega_0^2 - \omega^2}\right)$ is very small when $\omega \ll \omega_0$, and is π when $\omega \gg \omega_0$.

In real materials, there are many dipoles, and they oscillate at different ω_0. If there are N molecules per unit volume,

Oscillator strength

$$\tilde{P} = \frac{Ne^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}\right)\tilde{E}$$

Define $\tilde{\chi}_e$ as the complex susceptibility

$$\tilde{P} = \epsilon_0 \tilde{\chi}_e \tilde{E}$$

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\epsilon_r\vec{E} = \epsilon_0 E + \tilde{P} = \epsilon_0(1+\tilde{\chi}_e)E$$

$$\Rightarrow \epsilon_r(\omega) = 1 + \tilde{\chi}_e$$

$$= 1 + \frac{Ne^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

$$= \epsilon_r'(\omega) + i\epsilon''(\omega) = n^2(\omega)$$

Comments:

① Complex $\epsilon(\omega)$ is derived from classical harmonic oscillator model. f_j is phenomenological. According to QM, $\sum_j f_j = 1$

② Ordinarily, ϵ'' is negligible, when it close to resonances ω_j $\epsilon''(\omega)$ plays an important role.

So, what are the physical meaning of real/img. parts?

$$\nabla^2 \tilde{E} - \tilde{\epsilon}\mu_0 \frac{\partial^2 \tilde{E}}{\partial t^2} = 0 \quad \text{(wave eq. in dispersive media)}$$

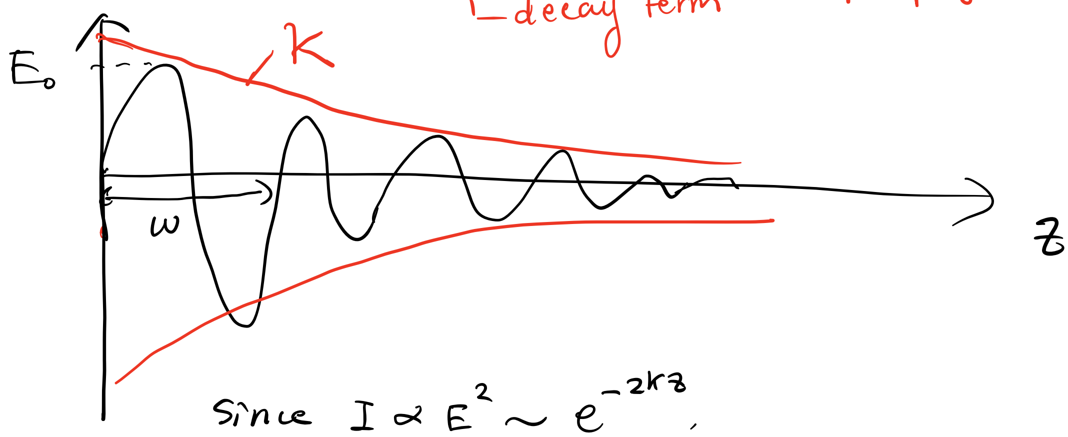
Plane wave solution:

$$\hat{\tilde{E}}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z - \omega t)} \quad ①$$

where $\tilde{k} = k + iK$, plug in ①,

$$\hat{E}(z,t) = \tilde{E}_0 e^{-Kz} e^{i(kz-\omega t)}$$

decay term propagation term



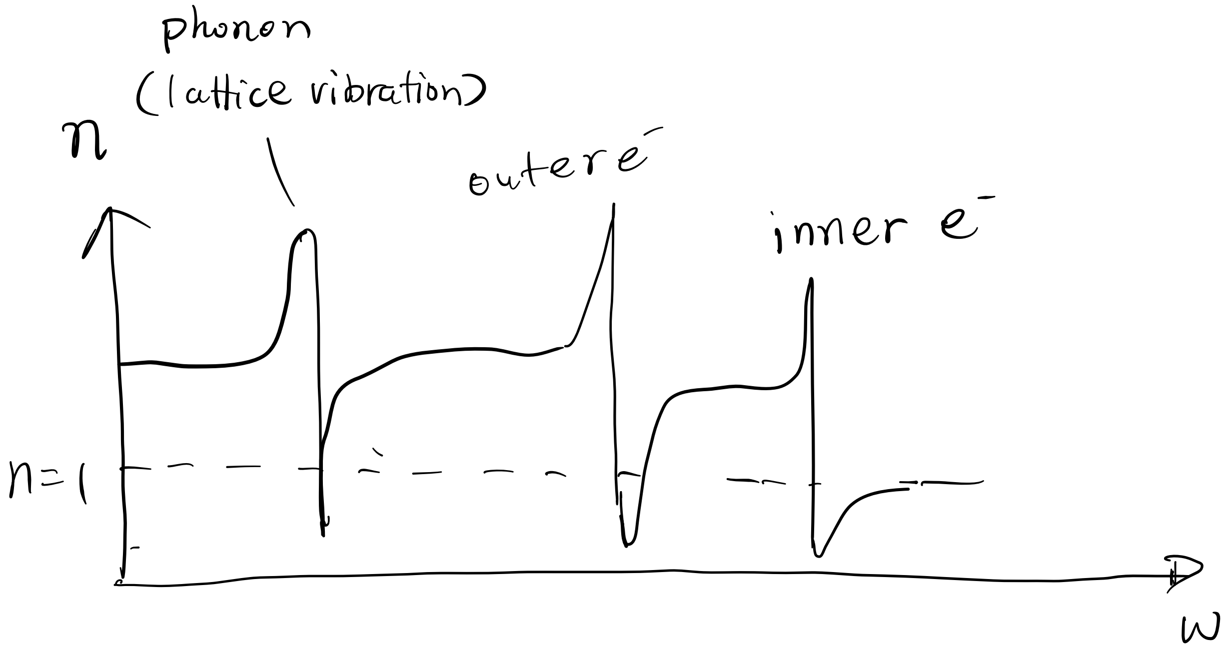
Since $I \propto E^2 \sim e^{-2Kz}$,

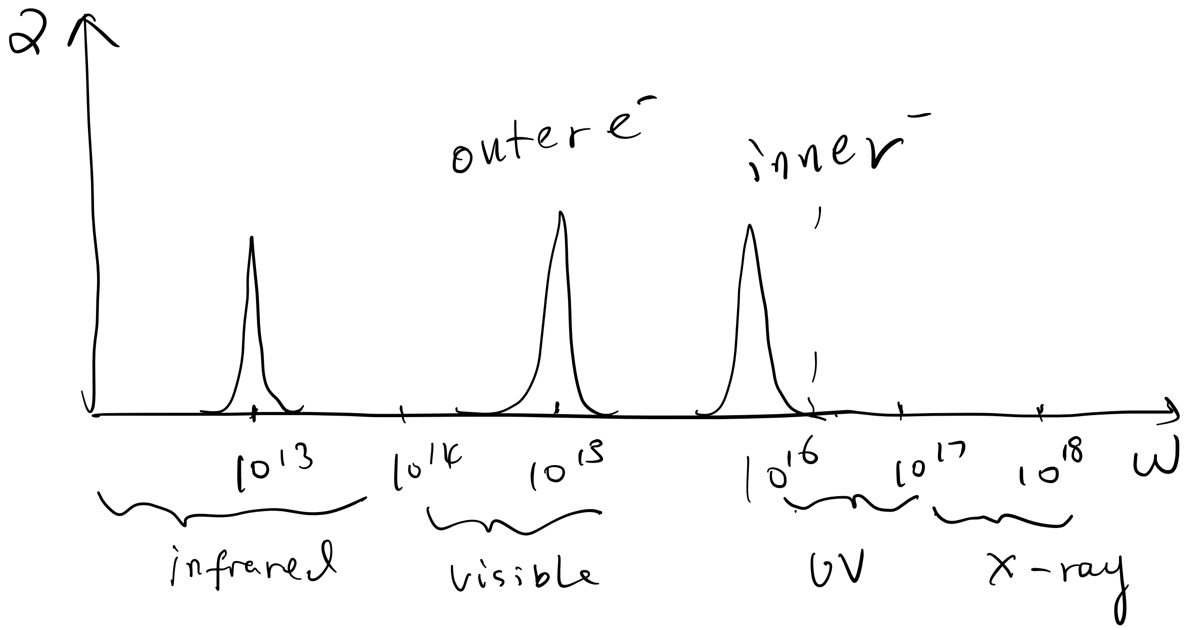
$\alpha = 2K$ is called absorption coefficient

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \simeq \frac{\omega}{c}\left(1 + \frac{1}{2}\epsilon\right) = \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega_0^2 - i\gamma_j\omega}\right]$$

$$n = \frac{ck}{\omega} \simeq 1 + \frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$

$$\alpha = 2\kappa \simeq \frac{Ne^2}{m\epsilon_0 c} \sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2) + \gamma_j^2 \omega^2}$$





Comments:

① At high ω ($\omega \gg \omega_0$), e^- are too sluggish to respond to driving $\bar{E}$-field, the medium has zero polarization so the dielectric constant is unity.

② Between each absorption band, the medium is transparent. (α is ~ 0, n is almost constant)

③ Most commonaly: $\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$

not wavevector!

$$\alpha = \frac{4\pi\lambda}{\kappa}.$$

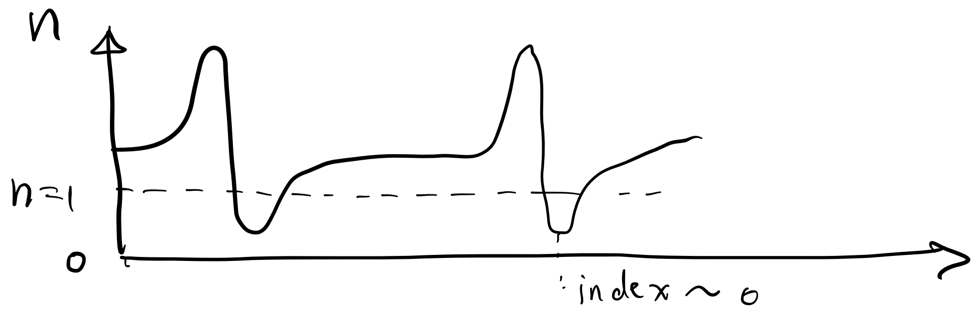
the real part of index captures the wave propagation (phase)

the imaginary part captures the loss

④ There are interesting features close to resonances. index can be either very high, or close to zero

Examples

1. Epsilon near zero (ENZ) or index near zero materials.



Applications:

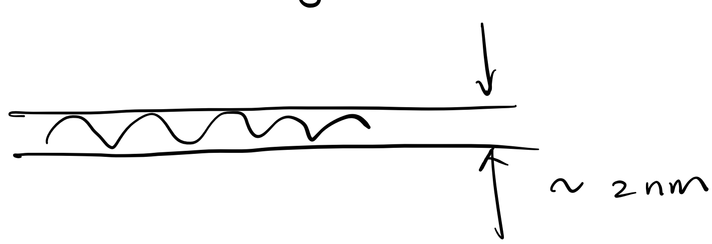
1. increase photon density of states
2. enhance spontaneous emission
3. $\phi = \omega t - kz = \omega t - n\frac{\omega}{c}\cdot z$

   no phase accumulated during propagatio
   (ENZ tunneling, Nader Engheta, Upenn)

Problems?

1. high reflection: $r = \frac{n-1}{n+1} \simeq 1$
2. high-loss.

2. Ultrathin waveguide (ultilize very large n around resonances)



(Ertugral Cubuken, Nature Nano, 2018)

2. Kramers-Kronig (K-K) relationship

$$\begin{cases} n(\omega)-1 = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \dfrac{\omega' k(\omega')}{\omega'^2-\omega^2}\, d\omega' \\ k(\omega) = -\dfrac{2}{\pi\omega} P \displaystyle\int_0^\infty \dfrac{\omega' [n(\omega')-1]}{\omega'^2-\omega^2}\, d\omega' \end{cases}$$

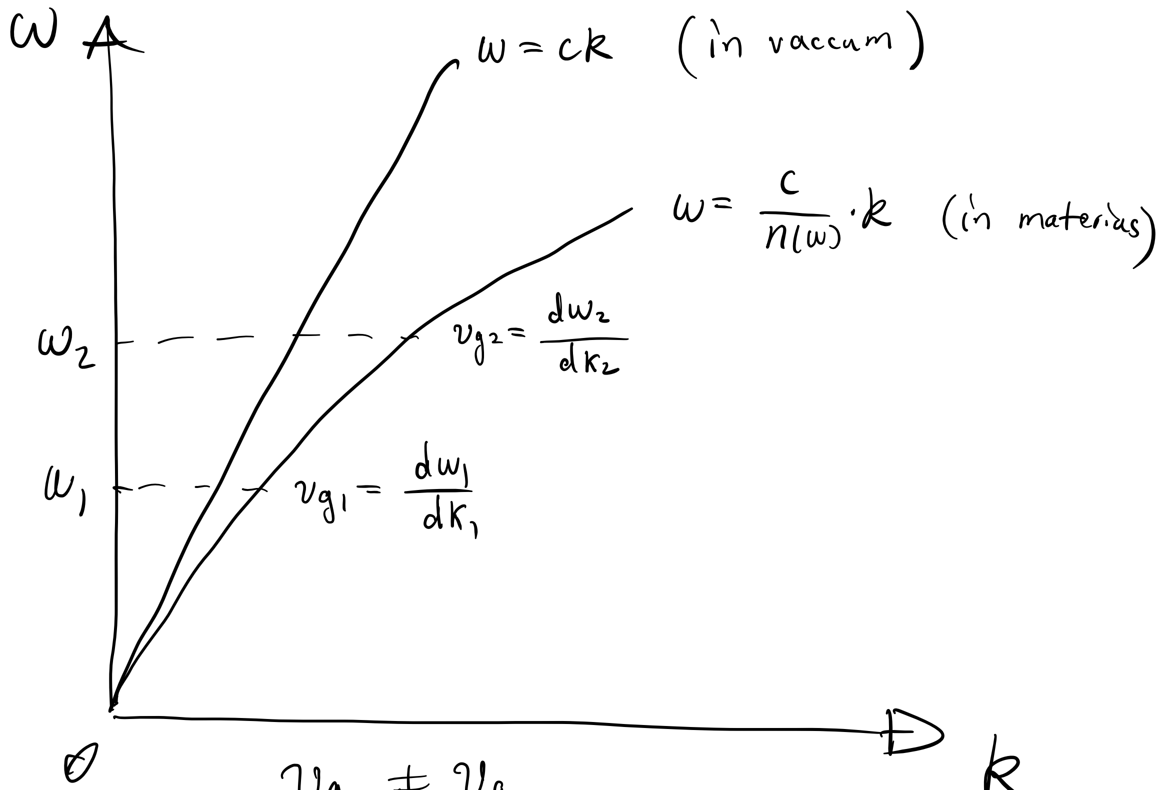
Phsical meaning: it allows us to calculate n from k, or vice versa.

Applications: Sellmeier eq.

$$\boxed{n^2 = 1 + \frac{A\lambda^2}{\lambda^2-\lambda_1^2} + \frac{B\lambda^2}{\lambda-\lambda_2^2} + \frac{C\lambda^2}{\lambda-\lambda_3^2}}$$

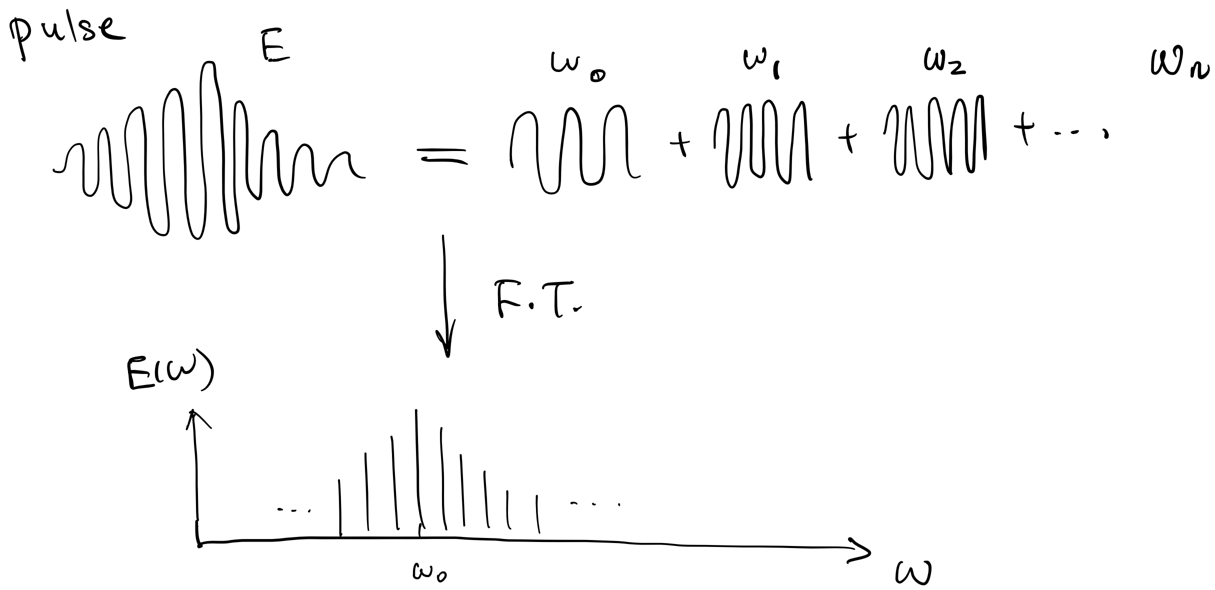
Very useful for modeling refractive index of various dielectric materials

3. Group velocity (P13, P315. Yariv)



$v_{g1} \neq v_{g2}$

For monochromatic plane wave, it doesn't matter much. For propagation of an optical pulses (which contain different ω), different frequency component of the pulse propagate with different speeds. This can lead to change of the shape or spreading of a pulse.

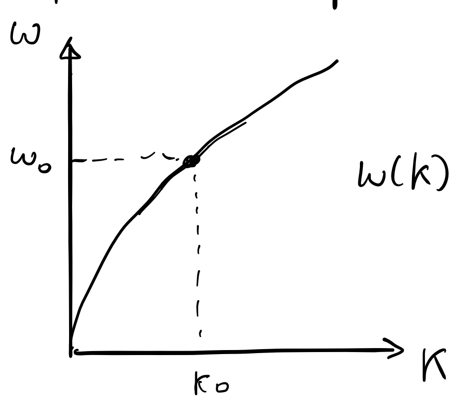


Electric-field of a pulse:

$$E(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i[\omega(k)t - kz]}\, dk \qquad ①$$

which is a sum of mononchromatic plane waves, and a solution of Maxwell equation

Expand the dispersion relation $\omega(k)$ by Taylor expansion



$$\omega(k) = \omega_0 + \left(\frac{d\omega}{dk}\right)_{\omega_0}(k-k_0) + \underbrace{\frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)(k-k_0)^2 + \cdots}_{\text{higher order term}}$$

plug into eq. ①, and ignore high order terms,

$$E(z,t) = e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} A(k) \exp\left\{ i\left[\left(\frac{d\omega}{dk}\right)_{\omega_0} t - z \right](k-k_0) \right\} dk$$

Define $V\left[z - \left(\frac{d\omega}{dk}\right)_{\omega_0} t\right]$ as envelop function

$$E(z,t) = e^{i(\omega_0 t - k_0 z)} V\left[z - \left(\frac{d\omega}{dk}\right)_{\omega_0} t\right]$$

The envelop travels along z with a velocity

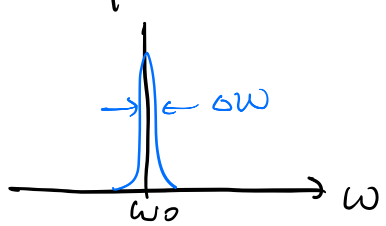
$$\boxed{v_g = \left(\frac{d\omega}{dk}\right)\bigg|_{\omega_0}} \quad \text{(group velocity of pulse)}$$

Comments:

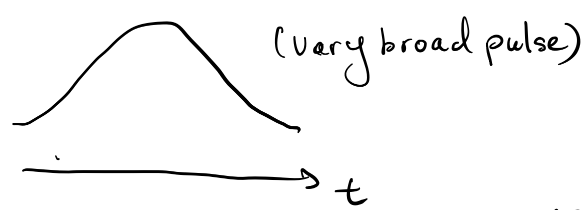
If we ignore the higher order terms (linear dispersion $\omega(k)$), the pulse envelop travels at v_g without distortion

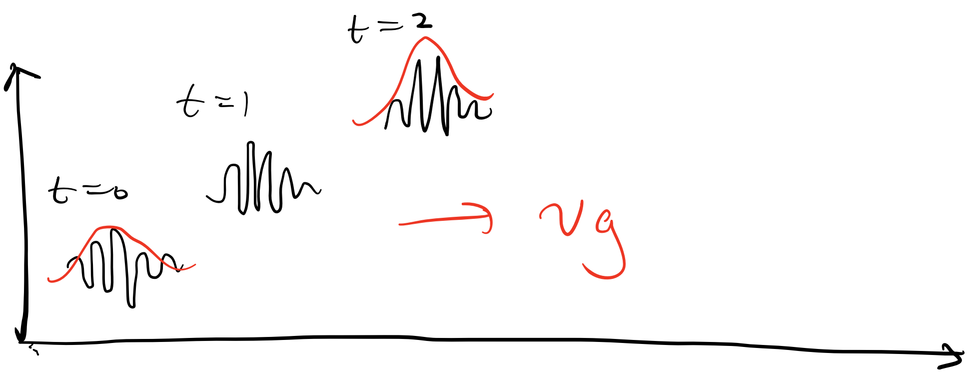
This linear assumption is only valid when $\omega \simeq \omega_0$ $(k - k_0 \simeq 0)$

Freq. domain



Time domain





In materials: $k = n(\omega)\dfrac{\omega}{c}$

$$\boxed{\begin{aligned} &\Rightarrow \text{phase velocity: } v_p = \frac{c}{n(\omega)} \\ &\text{group velocity: } v_g = \left(\frac{d\omega}{dk}\right)_{\omega_0} = \frac{c}{n + \omega\left(\frac{dn(\omega)}{d\omega}\right)} \\ &\qquad\text{or } = \frac{c}{n - \lambda\left(\frac{dn}{d\lambda}\right)} \end{aligned}}$$

Proof:

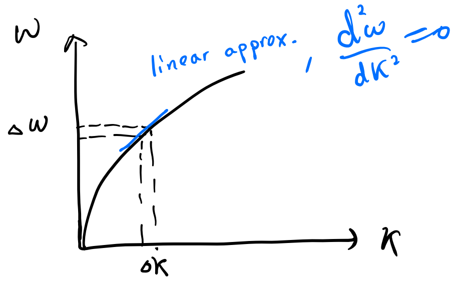
$$\frac{1}{v_g} = \left(\frac{dk}{d\omega}\right)_{\omega_0} = \frac{d}{d\omega}\left(\frac{n(\omega)\cdot\omega}{c}\right) = \frac{dn(\omega)}{d\omega}\cdot\frac{\omega}{c} + \frac{n}{c}$$

$$= \frac{1}{c}\left(n + \omega\left(\frac{dn(\omega)}{d\omega}\right)\right)$$

4. Group velocity dispersion (GVD)

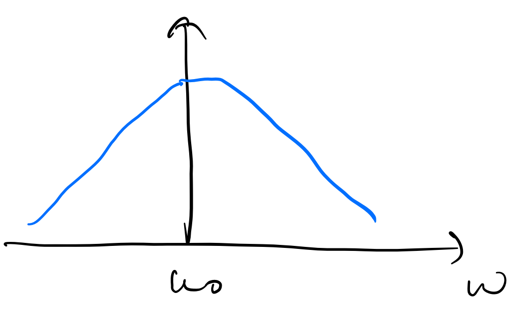
Pulse shape remains undistorted only when

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}(k-k_0) \text{ is valid. } (\Delta\omega \text{ is small})$$

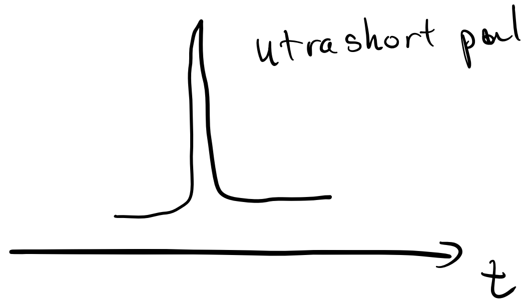


But when $\Delta\omega$ is large, and $(k-k_0)$ is not negligible, the term $\frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_{\omega_0}(k-k_0)^2$ is not negligible.

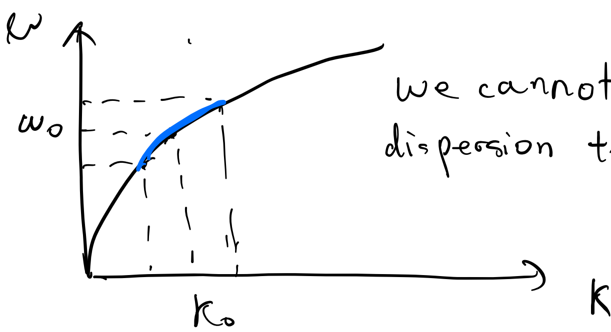
Freq. domain



Time domain



ultrashort pulses



we cannot use linear dispersion to approximate $\omega(k)$

Physical meaning:

v_g is not the same for each spectral component of the pulse. This is known as group velocity dispersion

Spread of group velocity:

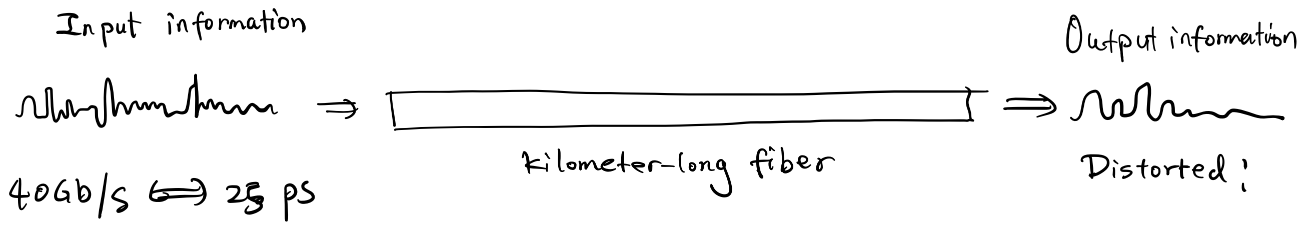
$$\Delta v_g = \left(\frac{d^2\omega}{dk^2}\right)_{\omega_0} \Delta k = \left(\frac{dv_g}{dk}\right)_{\omega_0} \cdot \Delta k$$

$\Delta v_g \cdot T$ is the pulse spread in position within the time of flight. T

This is a problem for

1. Fiber optical communications
2. Ultrafast optics.

Example: Fiber optical communication



Transmission time through a length L of fiber:

$$T = \frac{L}{v_g} = L\left(\frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega}\right) = L\left(\frac{n}{c} - \frac{\lambda}{c}\frac{dn}{d\lambda}\right)$$

Define Dispersion parameter: $D = \frac{1}{L}\frac{dT}{d\lambda}$,

$$D = -\frac{1}{c\lambda}\left(\lambda^2 \frac{d^2 n}{d\lambda^2}\right) = -\frac{2\pi c}{\lambda^2}\frac{dk^2}{d\omega^2} = -\frac{\lambda}{c}\frac{d^2 n}{d\lambda^2} \;(\text{ps/km}\cdot\text{nm})$$

where $\frac{d^2 k}{d\omega^2} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = \text{GVD} = \underline{\underline{\beta_2}} \quad (\text{fs}^2/\text{mm})$

$$\begin{cases} \beta_1 = \frac{dk}{d\omega} = \frac{1}{v_g} \\ \beta_2 = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) \end{cases}$$

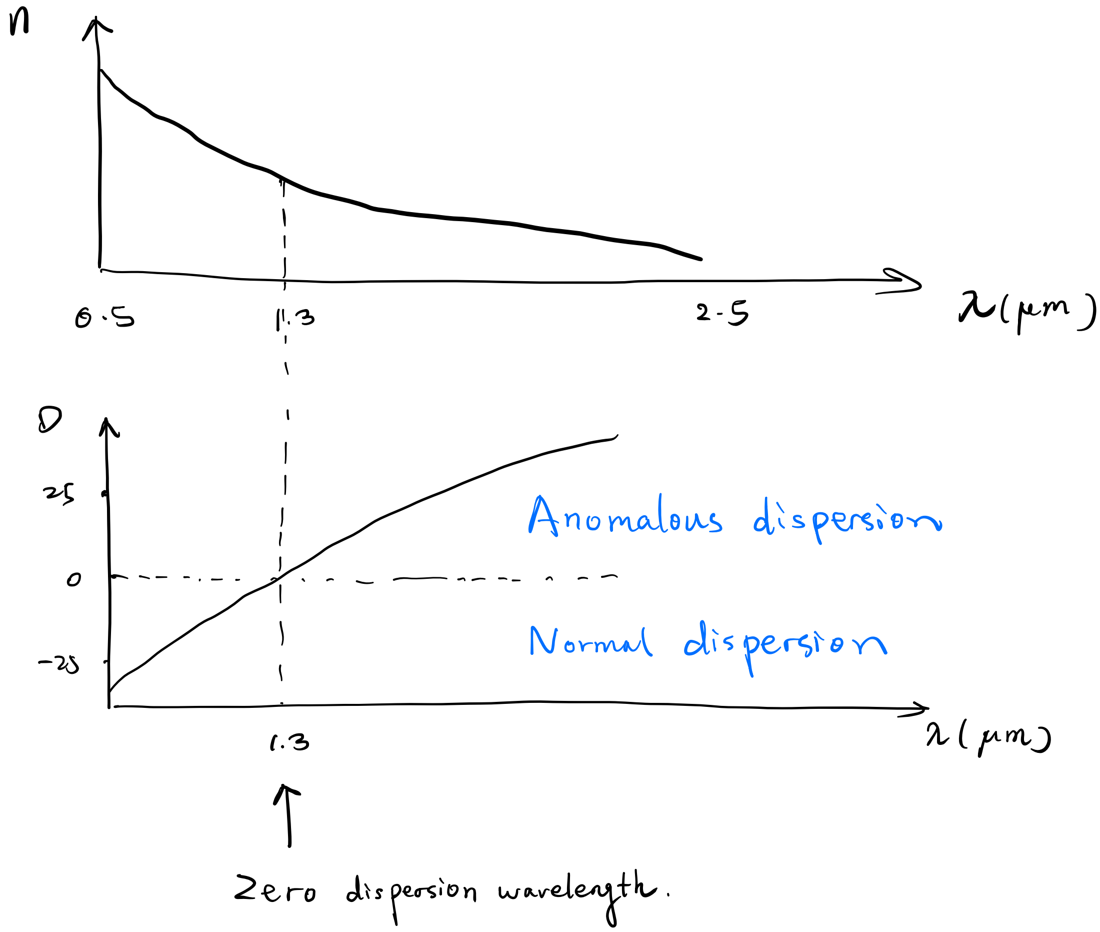
Comments:

① $\beta_2(\text{GVD}) \sim \frac{d^2 n}{d\lambda^2}$

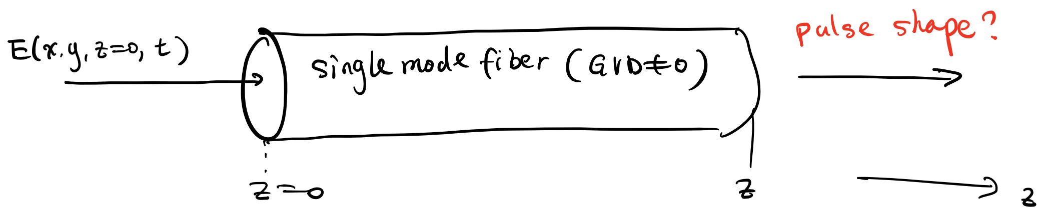
② $\frac{d^2 n}{d\lambda^2} > 0$, $\lambda\uparrow$ $v_g\downarrow$, GVD >0, $D<0$, Normal dispersion

③ $\frac{d^2 n}{d\lambda^2} < 0$, $\lambda\uparrow$, $v_g\uparrow$ GVD <0, $D>0$, Anomolous dispersion

Example: fused silica fiber



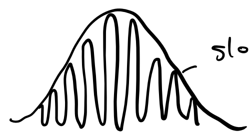
6. Optical pulse spreading (brodening) in dispersive media (Yariv, P317)



Input: Temporal Gaussian pulse:

$E(x,y,z=0,t) = u_0(x,y)\exp(-a t^2 + i\omega_0 t)$

Assuming slowly varing envelop: $\sqrt{a} \ll \omega_0$



Input pulse:

$$E(z=0,t) = \underbrace{e^{i\omega_0 t}}_{\text{carrier}} \cdot \underbrace{\int F(\Omega) e^{i\Omega t} d\Omega}_{\text{envelop}} = \int F(\Omega) e^{i(\omega_0+\Omega)t} \cdot d\Omega$$

Output pulse:

$$E(z,t) = \int F(\Omega) e^{i[(\omega_0+\Omega)t - \beta(\omega_0+\Omega)z]} \cdot d\Omega, \quad F(\Omega) = \sqrt{\frac{1}{4\pi a}} \exp\left(-\frac{\Omega^2}{4a}\right)$$

Expand $\beta(\omega_0+\Omega)$ the center frequency ω_0

$$\beta(\omega_0+\Omega) = \beta(\omega_0) + \left.\frac{d\beta}{d\omega}\right|_{\omega_0} \Omega + \frac{1}{2} \left.\frac{d^2\beta}{d\omega^2}\right|_{\omega_0} \Omega^2 + \cdots \qquad (2)$$

plug (2) into (1) and using $\beta_0 = \beta(\omega_0)$, $\left.\frac{d\beta}{d\omega}\right|_{\omega_0} = \frac{1}{v_g}$

$$E(z,t) = \exp\left[i(\omega_0 t - \beta_0 z)\right] \int_{-\infty}^{\infty} d\Omega\, F(\Omega) \exp\left\{ i\left[\Omega t - \frac{\Omega z}{v_g} - \frac{1}{2}\frac{d}{d\omega}\left(\frac{1}{v_g}\right)\Omega^2 z\right]\right\} \quad (3)$$

$$\equiv \exp\left[i(\omega_0 t - \beta_0 z)\right] \cdot \underbrace{\mathcal{E}(z,t)}_{\text{Envelop function}}$$

Comments

① here we ignore higher order terms Ω^3 and ...

② when GVD $=0$, $\frac{d}{d\omega}\left(\frac{1}{v_g}\right) = 0$,

$$\mathcal{E}(z,t) = \int_{-\infty}^{\infty} d\Omega\, F(\Omega) \exp\left\{ i\Omega\left(t - \frac{z}{v_g}\right)\right\}$$

$$= \mathcal{E}\left[0, \left(t - \frac{z}{v_g}\right)\right]$$

pulse envelop remains unchanged and propagates at v_g.

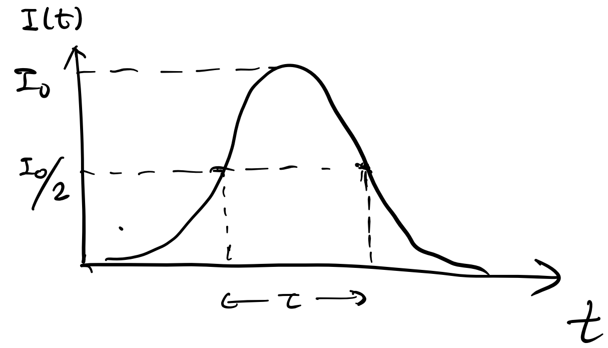
Generic case:

plug in $F(\Omega) = \sqrt{\frac{1}{4\pi\alpha}} \exp\left(-\frac{\Omega^2}{4\alpha}\right)$, and using $a = \left(\frac{d^2\beta}{d\omega^2}\right)/2$

$$\mathcal{E}(z,t) = \sqrt{\frac{1}{4\pi\alpha}} \cdot \int_{-\infty}^{\infty} \exp\left\{-\left[\Omega^2\left(\frac{1}{4\alpha} + iaz\right) - i\left(t - \frac{z}{v_g}\right)\Omega\right]\right\} d\Omega$$

$$= \frac{1}{\sqrt{1 + i4a\alpha z}} \exp\left(-\frac{\left(t - \frac{z}{v_g}\right)^2}{1/\alpha + 16a^2z^2\alpha}\right) \exp\left(i\, \frac{4az\left(t - z/v_g\right)^2}{1/\alpha^2 + 16a^2z^2}\right)$$

Pulse duration (τ) : (FWHM of the intensity profile $E^2(z,t)$)
(width)



At the output:

$$\tau(z) = \sqrt{2\ln 2}\sqrt{\frac{1}{a} + 16a^2z^2}$$

Initial pulse width:

$$\tau_0 = \tau(z=0) = \sqrt{\frac{2\ln 2}{a}}$$

The pulse width after propagating a distance L

$$\tau(L) = \tau_0\sqrt{1 + \left(\frac{8aL\ln 2}{\tau_0^2}\right)^2}$$

At larger distances ($|aL| \gg \tau_0^2$),

$$\tau(L) \simeq \frac{(8\ln 2)aL}{\tau_0} = \frac{4\ln 2}{v_g^2}\left|\frac{dv_g}{d\omega}\right|\frac{L}{\tau_0}$$

Recall $D = -\frac{2\pi c}{\lambda^2}\left(\frac{d^2\beta}{d\omega^2}\right)$

$$\tau(L) = \tau_0\sqrt{1 + \left(\frac{2\ln 2}{\pi c}\frac{DL\lambda^2}{\tau_0^2}\right)^2}$$

if D: ps/km·nm, λ: µm, L: km. τ_0: ps

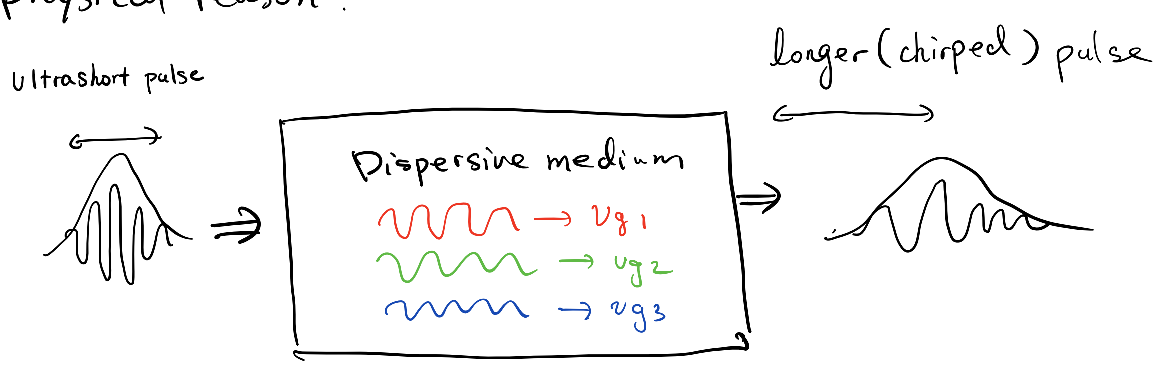
$$\tau(L) = \tau_0 \sqrt{1 + \left(\frac{1.47 D L \lambda^2}{\tau_0^2}\right)^2}$$

Comments:

① $D = 0$, $\tau(L) = \tau_0$

② Non-zero GVD (or D) will broaden the pulse

Physical reason?



③ pulse spreading (broadening) also depends on input pulse width (τ_0) $\longrightarrow$ RP photonics

④ pulse broadening is a linear optical effect due to GVD $\left(\frac{d^2\beta}{d\omega^2} \neq 0\right)$.

7. Frequency chirp.

Just now, we discussed the envelop function $\mathcal{E}(z,t)$

Let's write down the whole output field:

$$E(z,t) = \frac{1}{\sqrt{1+i4a\alpha z}} \exp\left(-\frac{(t-z/v_g)^2}{1/\alpha + 16a^2 z^2 \alpha}\right) \exp\left(i(\omega_0 t - \beta_0 z) + i\underbrace{\frac{4az(t-\frac{z}{v_g})^2}{1/\alpha^2 + 16a^2 z^2}}_{\text{qudratic temporal phase}}\right)$$

where $a = \frac{1}{2}\frac{d^2\beta}{d\omega^2} = -\frac{1}{2v_g^2}\frac{dv_g}{d\omega}$

phase of the output field:

$$\phi(z,t) = \omega_0 t - \beta_0 z + \frac{4az(t-\frac{z}{v_g})^2}{1/\alpha^2 + 16a^2 z^2}$$

Recall the frequency of wave $\left(\phi = \omega t - kz, \quad \omega = \frac{\partial \phi}{\partial t}\right)$

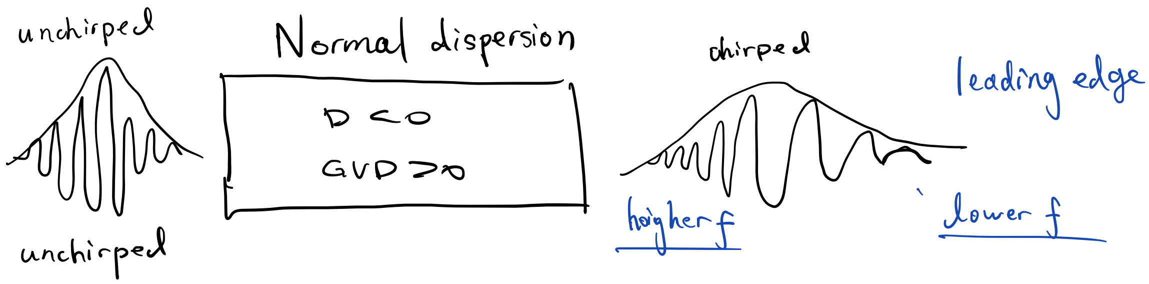
$$\underset{\text{(instantaneous freq.)}}{\omega(z,t)} = \frac{\partial}{\partial t}\phi(z,t) = \omega_0 + 8\frac{az}{(1/\alpha^2 + 16a^2z^2)}\underbrace{(t-\frac{z}{v_g})}_{\text{time-dependent}}$$

Comment:

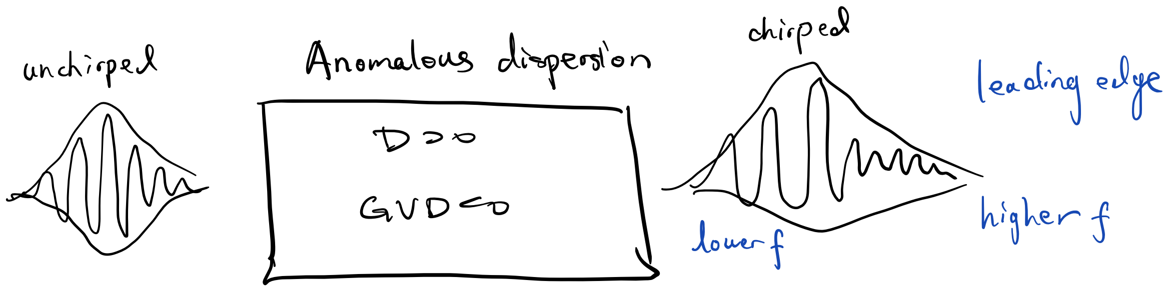
① In addition to pulse broadening, dispersive medium (GVD $\neq 0$) modifies the optical frequency.

② The frequency of output pulse is not a constant, but is linearly "chirped". This is because different "groups" of frequencies travel at different velocities.

③ Chirps can also be induced by nonlinear effects (Kerr effect...)



lower freq. components travel faster.



higher freq. components travel faster

· Dispersion compensation (pulse shaping / linear pulse compression)