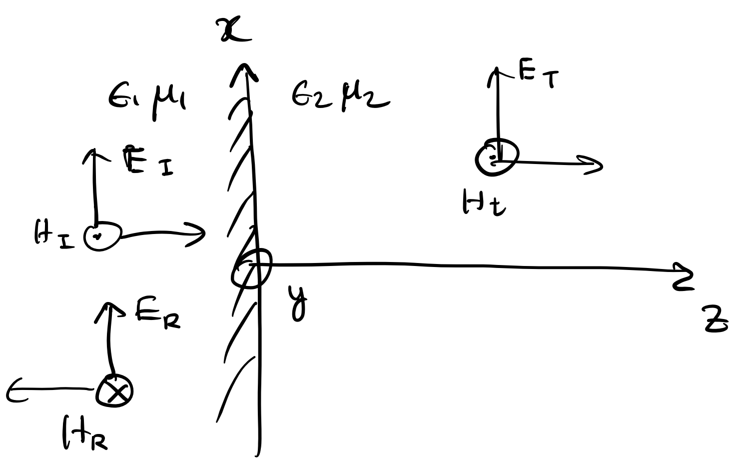
Lecture 3. Reflection and Refraction

Learning objectives:

1. Normal incidence
2. Oblique incidence
3. Snell and Fresnell's laws
4. Brewster angle,
5. Total internal reflection and critical angle
6. Evanescent field.

1. Normal incidence



Incident waves

$$\begin{cases} E_I(z,t) = E_0 e^{i(k_1 z - \omega t)} \hat{x} \\ H_I(z,t) = \frac{1}{\eta_1} E_0 e^{i(k_1 z - \omega t)} \hat{y} \end{cases}$$

Reflected waves

$$\begin{cases} E_R(z,t) = r E_0 e^{i(-k_1 z - \omega t)} \hat{x} \\ H_R(z,t) = -\frac{r}{\eta_2} E_{0R} e^{i(-k_1 z - \omega t)} \hat{y} \end{cases}$$

Transmitted waves

$$\begin{cases} E_T(z,t) = t E_0 e^{i(k_2 z - \omega t)} \hat{x} \\ H_T(z,t) = \frac{t}{\eta_2} E_{0T} e^{i(k_2 x - \omega t)} \hat{y} \end{cases}$$

B.C. tangential component of $\vec{E}$. $\vec{H}$ are contirous.

$$\left.\begin{aligned} 1 + r &= t \\ \frac{1}{\eta_1} - \frac{r}{\eta_2} &= \frac{t}{\eta_2} \end{aligned}\right\}$$

$\Rightarrow$

Reflection / Transmission Coefficients

$$r = \frac{E_R}{E_I} = -\frac{H_R}{H_I} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$t = \frac{E_T}{E_I} = \frac{\eta_2 H_T}{\eta_1 H_I} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

Since $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$, $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ $\quad n = \sqrt{\epsilon_r}$

$$\begin{cases} r = \dfrac{n_1 - n_2}{n_1 + n_2} \\[2ex] t = \dfrac{2n_1}{n_1 + n_2} \end{cases},$$

$$\text{Reflection } (R) = \frac{P_R}{P_I} \sim \frac{(E_R)^2}{(E_I)^2} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$\text{Transmission } (T) = \frac{P_T}{P_I} = \frac{\frac{1}{2}\epsilon_2 v_2}{\frac{1}{2}\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4n_1 n_2}{(n_1+n_2)^2}$$

$$r + t \neq 1 \;;\; R + T = 1$$

Example :

Air / glass interface

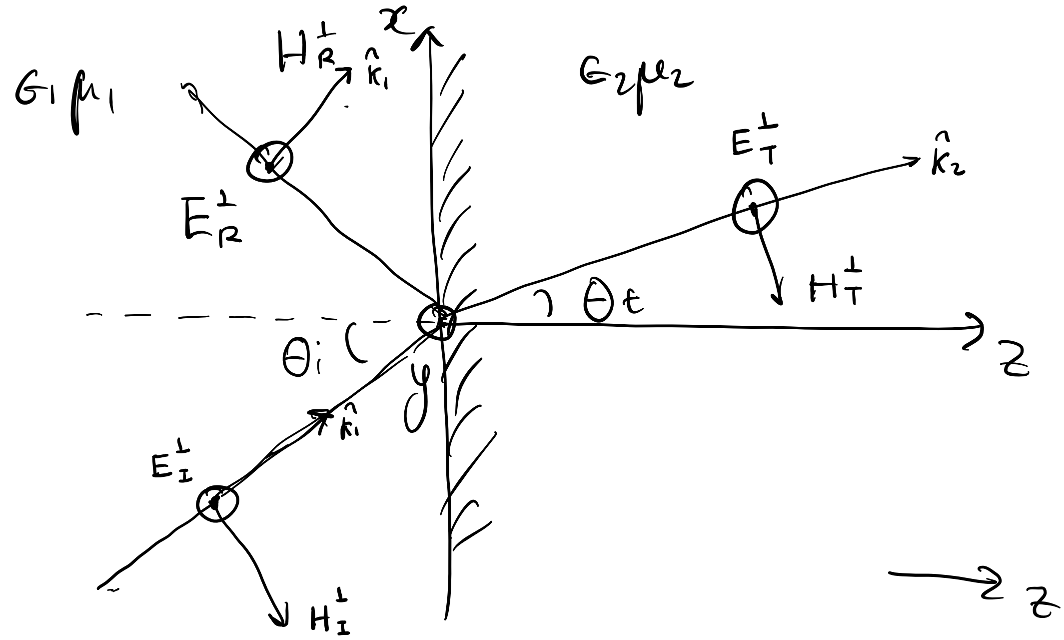
$n_{air} = 1$, $n_{glass} = 1.5$

$R = 4\%$, $T = 96\%$

2. Oblique Incidence

① Perpendicular polarization (S-polarized)

($\vec{E}$ is perpendicular to the plane of incidence)



Incidence wave

$$\begin{cases} E_I^\perp = \hat{a}_y E_0 e^{i[wt - k_1(x\sin\theta_i + z\cos\theta_t)]} \\ H_I^\perp = (-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)\dfrac{E_0}{\eta_1} e^{i[wt - k_1(x\sin\theta_i + z\cos\theta_t)]} \end{cases}$$

Reflected wave

$$\begin{cases} E_R^\perp = \hat{a}_y r_\perp E_0 e^{i[wt - k_1(x\sin\theta_r - z\cos\theta_r)]} \\ H_R^\perp = (\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r)\dfrac{r_\perp E_0}{\eta_1} e^{i[wt - k_1(x\sin\theta_r - z\cos\theta_r)]} \end{cases}$$

Transmitted wave

$$\begin{cases} E_T^{\perp} = \hat{a}_y t_{\perp} E_0 e^{i[\omega t - k(x\sin\theta_t + z\cos\theta_t)]} \\ H_T^{\perp} = (-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_T)\dfrac{t_{\perp} E_0}{\eta_2} e^{i[\omega t - k_2(x\sin\theta_t + z\cos\theta_t)]} \end{cases}$$

B.C.

$$\begin{cases} (E_I^{\perp} + E_R^{\perp})\Big|_{\substack{tan \\ z=0}} = (E_T^{\perp})\Big|_{\substack{tan \\ z=0}} \\ (H_I^{\perp} + H_R^{\perp})\Big|_{\substack{tan \\ z=0}} = (H_T^{\perp})\Big|_{\substack{tan \\ z=0}} \end{cases}$$

$$\Rightarrow \begin{cases} e^{-ik_1 x\sin\theta_t} + r_{\perp} e^{-ik_1 x\sin\theta_r} = t_{\perp} e^{-ik_2 x\sin\theta_t} \\ \dfrac{1}{\eta_1}\left(-\cos\theta_t e^{-ik_1 x\sin\theta_i} + r_{\perp}\cos\theta_r e^{-ik_1 x\sin\theta_r}\right) = -\dfrac{t_{\perp}}{\eta_2}\cos\theta_t e^{-ik_2 x\sin\theta_t} \end{cases}$$

$$\Rightarrow \begin{cases} \theta_i = \theta_r & \text{(Snell's Law of reflection)} \\ k_1\sin\theta_i = k_2\sin\theta_t & \text{(Snell's Law of refraction)} \end{cases}$$

$$\Rightarrow r_{\perp} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$t_{\perp} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

For most dielectric materials (non-Ferromagnetic)

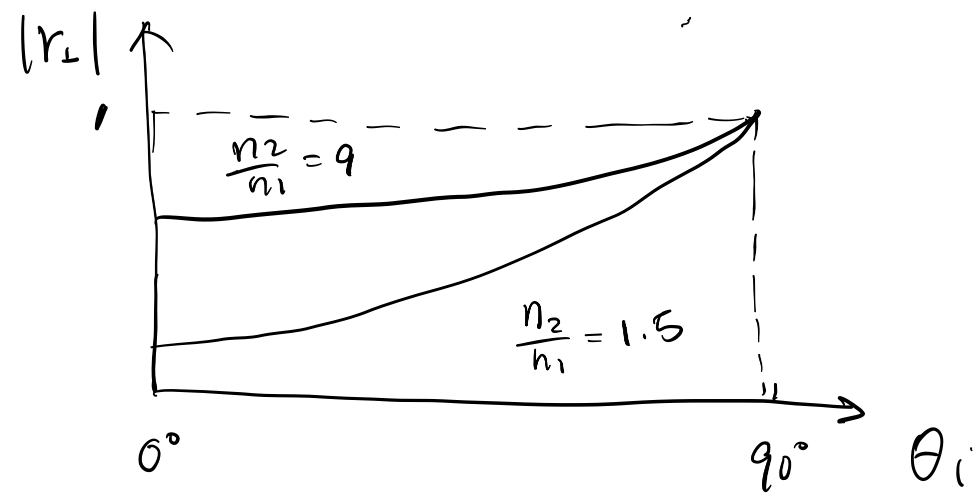
$\mu_1 = \mu_2 = \mu_0$

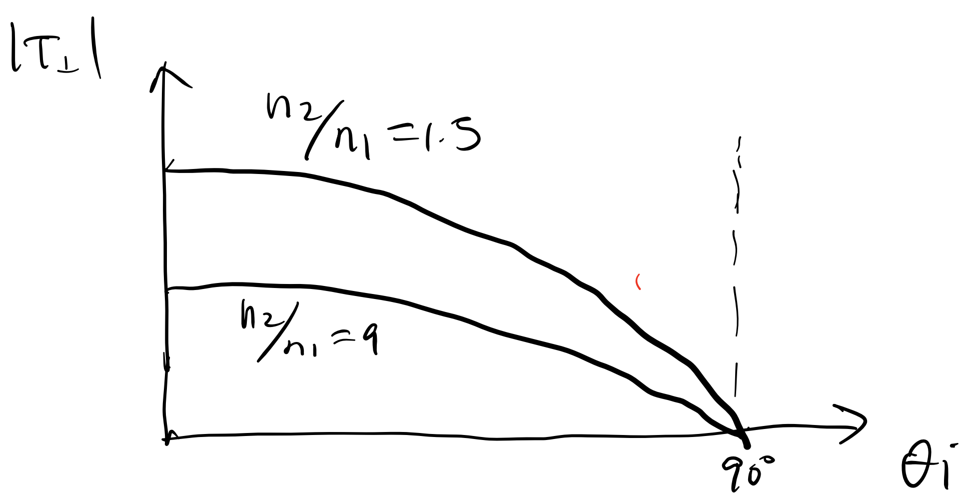
$$\boxed{\Rightarrow \quad r_\perp = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \qquad t_\perp = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}}$$

Fresnel's equations

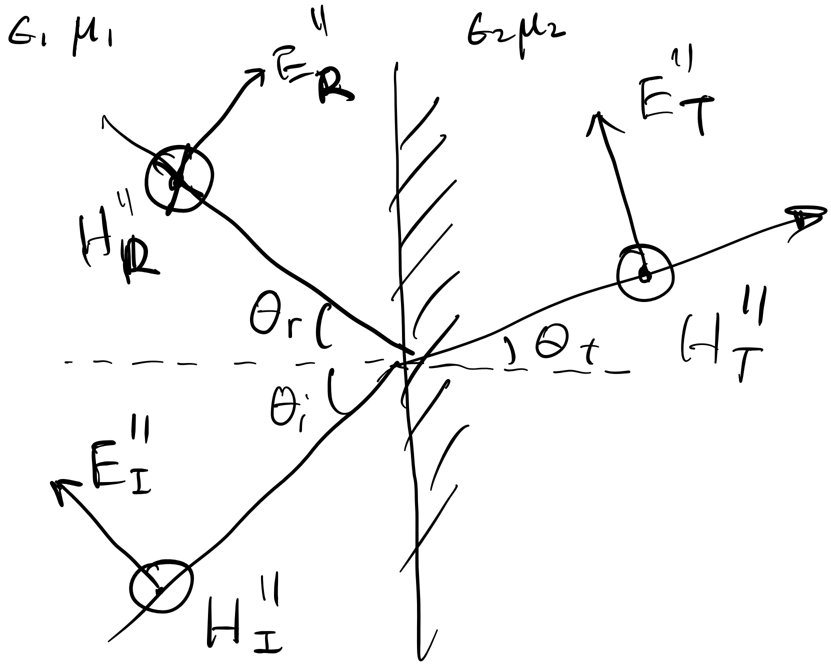
$n_1 \cos\theta_1 < n_2 \cos\theta_2$

For S-polarized light





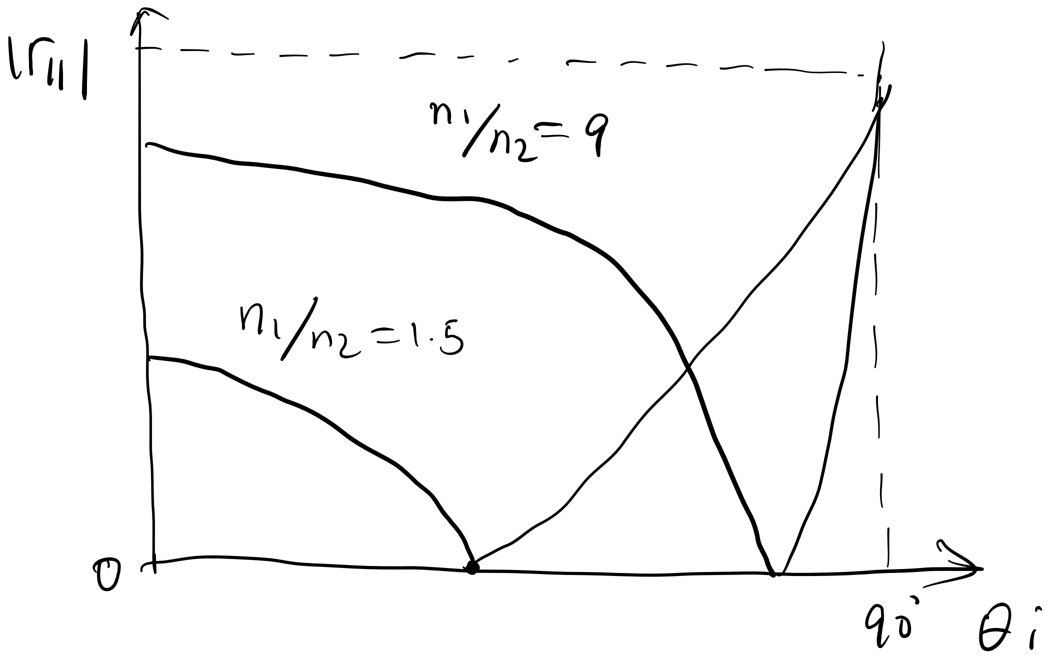
② Parallel polarization (p-polarized)

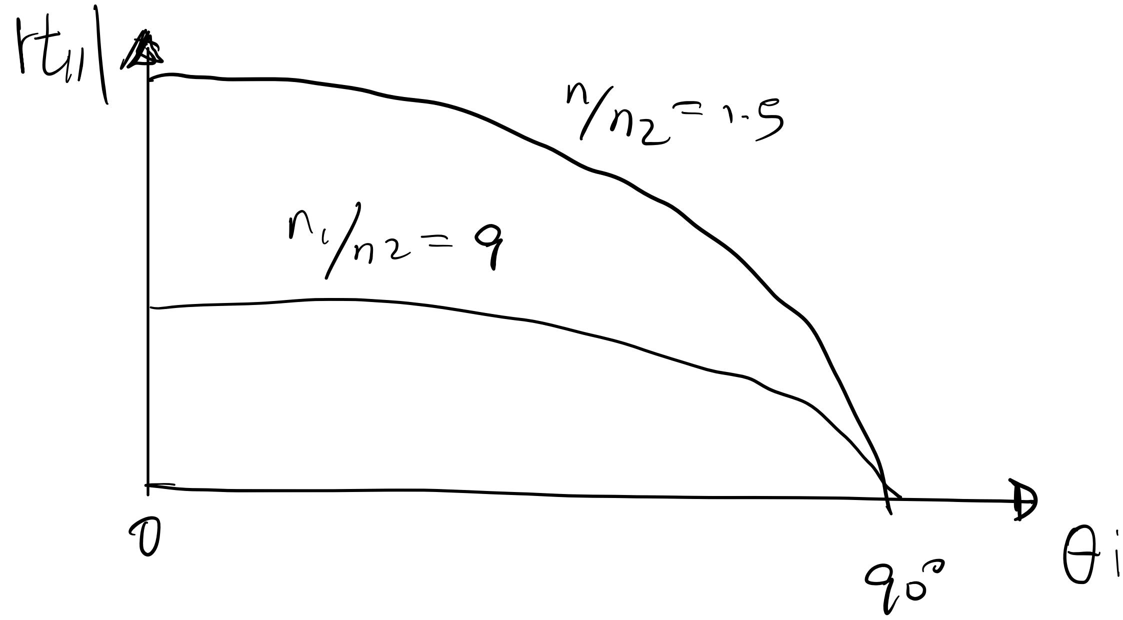


Fresnel's equations

$$\boxed{r_{\parallel} = \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i} \qquad t_{\parallel} = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i}}$$

$n_1 \cos\theta_t$ can equal to $n_2 \cos\theta_i$

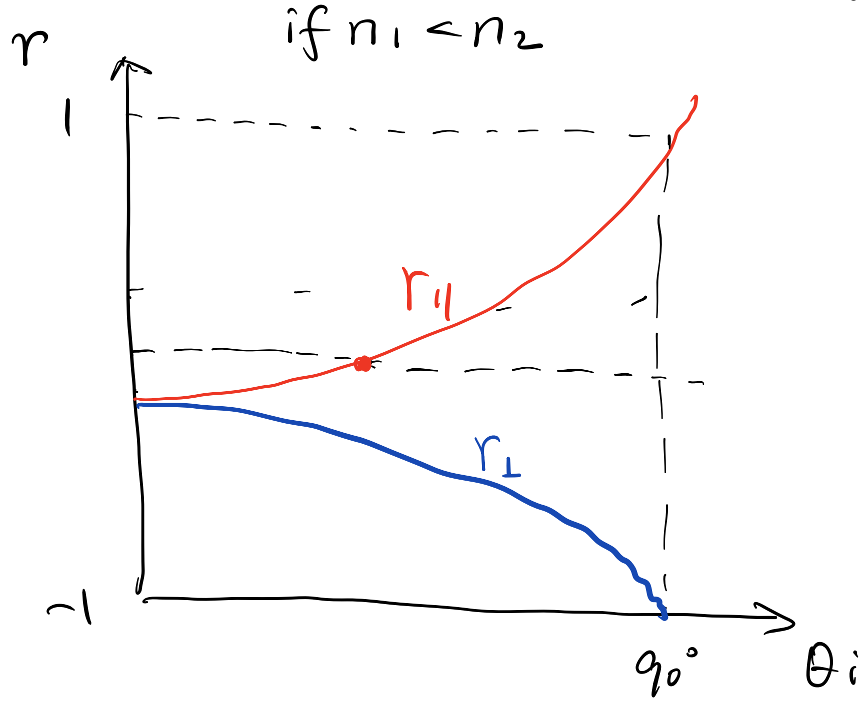




3. Total transmission (Brewster angle)

Plotting p- and s- polarized light together

if $n_1 < n_2$



Comments:

(normal inc.)

① Two polarizations are indistinguishable at $\theta_i = 0$

② Reflection = 1 at 90° for both polarizations

③ $r_{\parallel}$ becomes zero at Brewster angle,

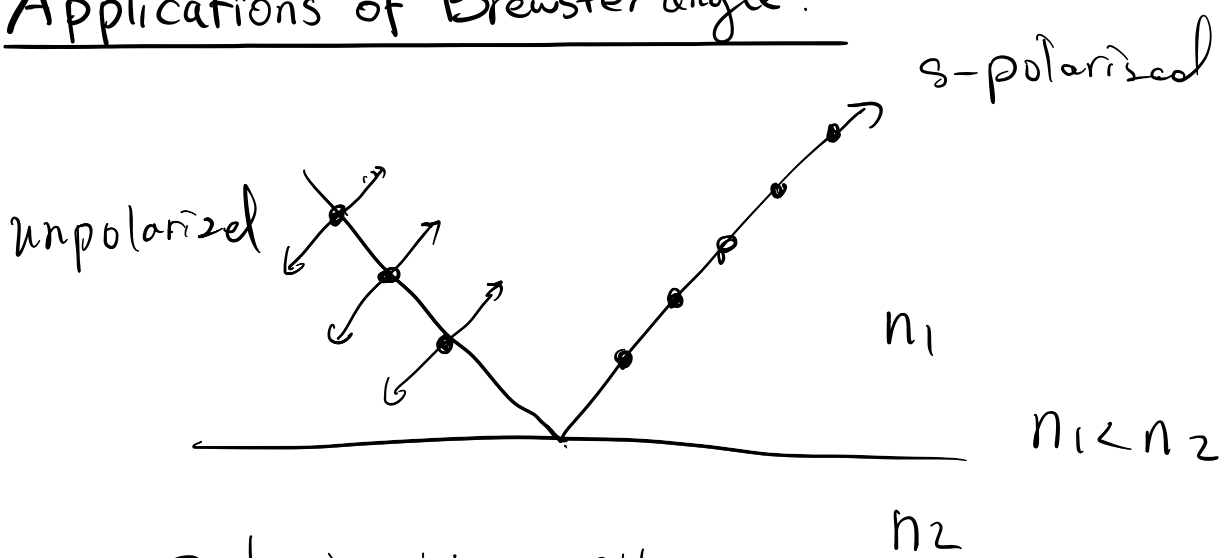
$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right)$$

eg. For air to glass, $n_{glass} = 1.5$,

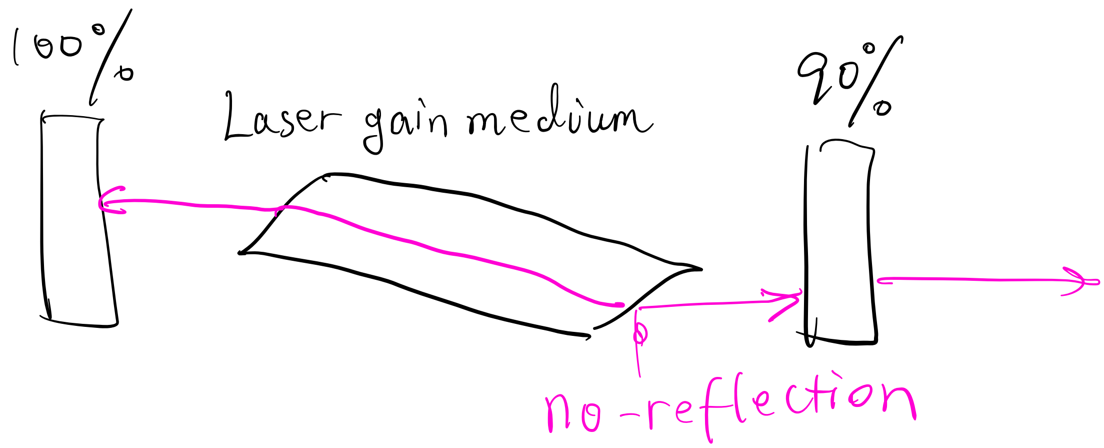
$$\theta_B = 56.3°$$

④ Total transmission only occurs at p-polarization

Applications of Brewster angle.



Polarization filter



4. Total internal Reflection — Critical angle

Is there an incident angle that allows the total reflection?

$$|r_\perp| = \frac{|n_1 \cos\theta_i - n_2 \cos\theta_t|}{|n_1 \cos\theta_i + n_2 \cos\theta_t|} = 1$$

This is satisfied when $n_2 \cos\theta_t$ is purely imaginary!

since $\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$

According to Snell's law,

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

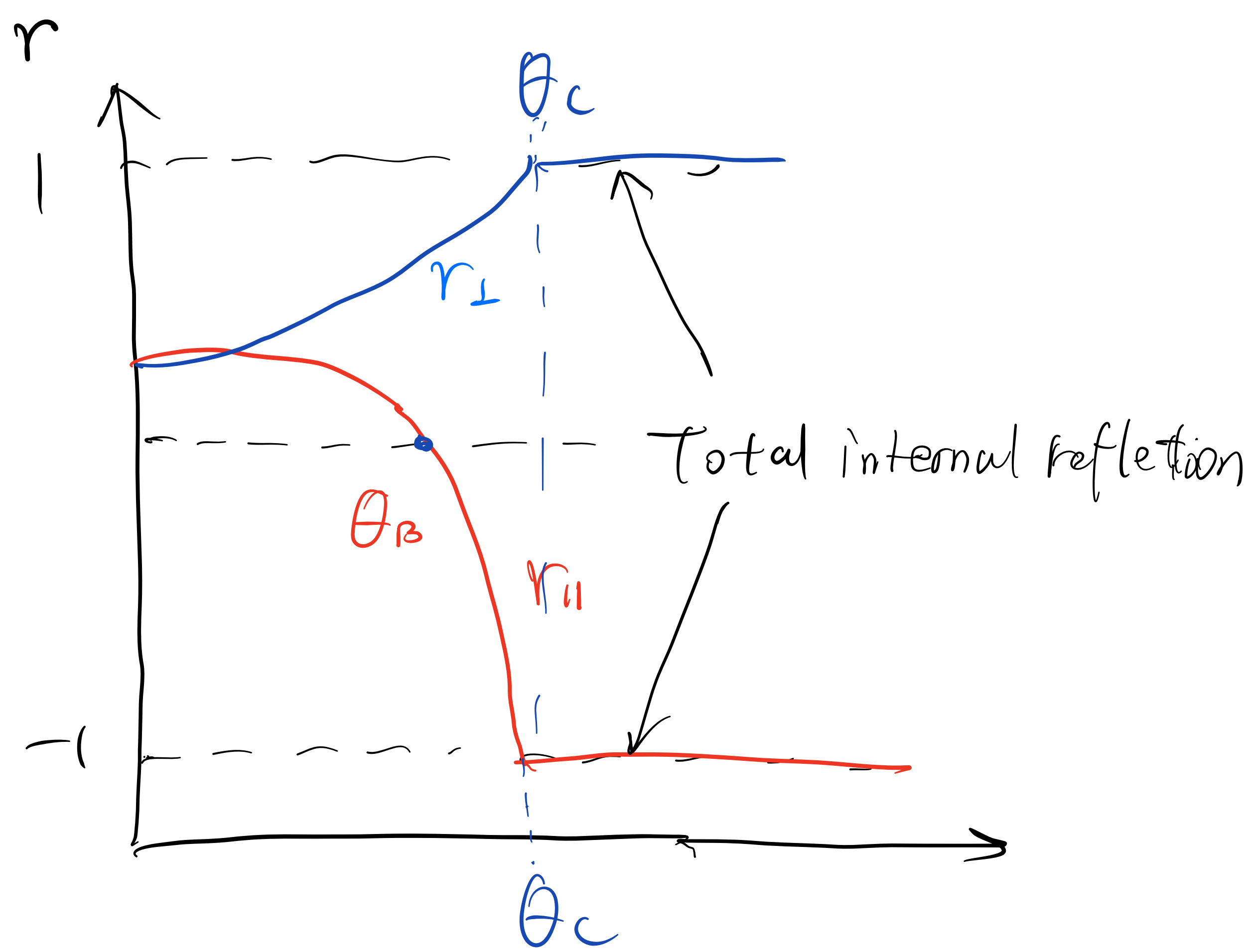
$$\Rightarrow \cos\theta_t = \sqrt{1 - \left(\frac{n_1}{n_2} \sin\theta_i\right)^2}$$

when $\theta_i > \arcsin\left(\frac{n_2}{n_1}\right)$, $|r| = 1$

when $n_2 > n_1$ there is no such angle

but when $n_2 < n_1$ this angle can be found!

Total internal reflection happens only when the light ray passes through the more dense optical medium to less dense optical medium



Air-glass interface

$$\theta_c = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.8^\circ$$

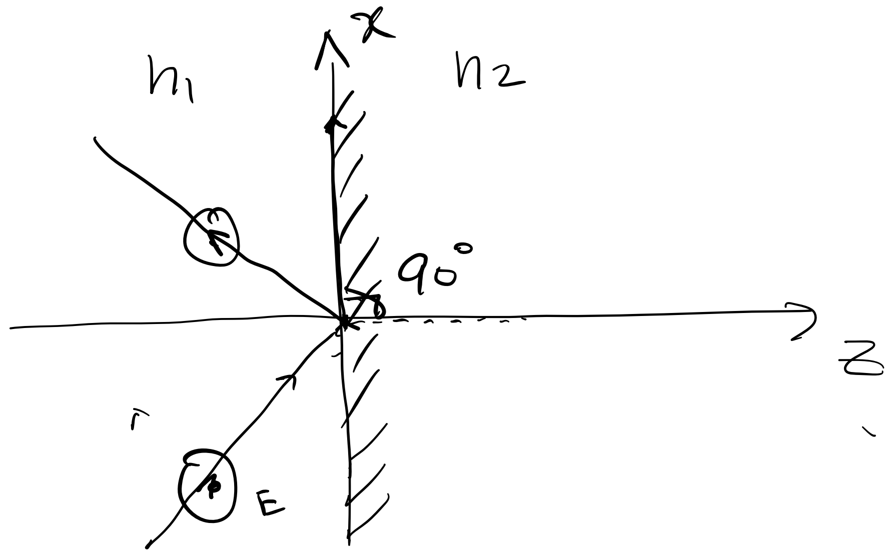
5. Evenescent Wave

The next question is : what happens to the angle of refraction and to the propagation of the wave when $\theta_i \geq \theta_c$?

① when $\theta_i = \theta_c$

According to Snell's Law,

$$\theta_t = \sin^{-1}\sqrt{\frac{n_1}{n_2}\sin\theta_i} = \sin^{-1}\sqrt{\frac{n_1}{n_2}\cdot\frac{n_2}{n_1}} = \sin^{-1}(-1) = 90^\circ$$



Also, $r_\perp \big|_{\theta_t = \theta_c} = 1$

$t_\perp \big|_{\theta_t = \theta_c} = 2$

Transmitted fields for s-polarized

$$\bar{E}_{\perp}^{t} = \hat{a}_y t_{\perp} E_o e^{i[\omega t - k_2(x\sin\theta_i + z\cos\theta_t)]}$$

$$H_{\perp}^{t} = (-\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t)\frac{t_{\perp} E_o}{\eta_2} e^{i[\omega t - k_2(x\sin\theta_t + z\cos\theta_t)]}$$

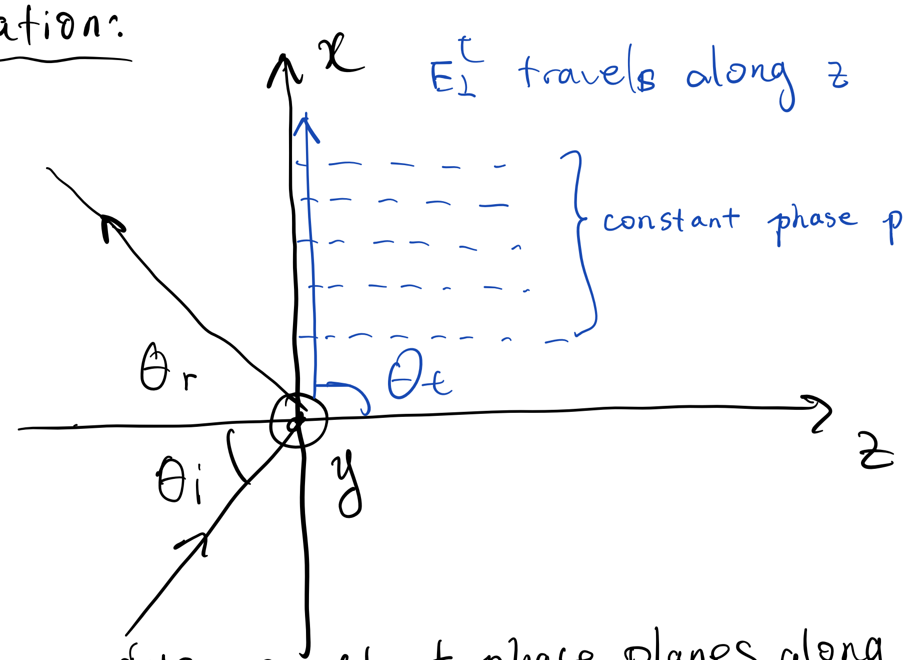
with $\theta_t = 90°$

$$E_{\perp}^{t} = \hat{a}_y \, 2E_o e^{i(\omega t - k_2 x)}$$

$$H_{\perp}^{t} = \hat{a}_z \frac{2E_o}{\eta_2} e^{i(\omega t - k_2 x)}$$

which represent a plane wave that travels parallel to the interface in the $\hat{x}$ direction

Illustration:

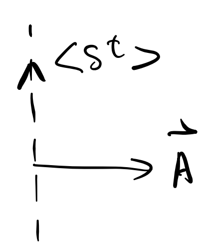


Surface wave, constant phase planes along $\hat{z}$

The average power density of this transmitted wave is

$$\langle \vec{S}^t \rangle \Big|_{\theta_t=\theta_c} = \frac{1}{2}\left(\vec{E}_\perp^t \times H_\perp^t\right)\Big|_{\theta_t=\theta_c} = \hat{a}_x \frac{2|E_0|^2}{\eta_2}$$

Physical meaning: no power transfer across the interface!
since $\langle \vec{S}^t \rangle \cdot \vec{A} = 0$



Also

$$\left|\langle \vec{S}^i \rangle\right|_{\theta_i=\theta_c} = \left|\frac{1}{2}\mathrm{Re}\left(E_\perp^i \times H_\perp^{i*}\right)\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}\left|\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right| = \frac{|E_0|^2}{2\eta_1}$$

$$\left|\langle S^t \rangle\right|_{\theta_i=\theta_c} = \left|\frac{1}{2}\mathrm{Re}\left(E_\perp^r \times H_\perp^{r*}\right)\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}\left|\hat{a}_x \sin\theta_i - \hat{a}_z \cos\theta_i\right| = \frac{|E_0|^2}{2\eta_1}$$

So $\left|\langle S^i \rangle\right| = \left|\langle S^t \rangle\right|$, 100% power reflection

② when $\theta_i > \theta_c$

Snell's Law:

$$\sin\theta_t\Big|_{\theta_i>\theta_c} = \frac{n_1}{n_2}\sin\theta_i\Big|_{\theta_i>\theta_c} > 1$$

this can only be satisfied when θ_t is complex!

Therefore, for $\theta_i > \theta_c$, there is no physically realizable θ_t!

Let's examine the field in medium 2

$$E_\perp^t = \hat{a}_y t_\perp E_0 e^{i[\omega t - k_2(x\sin\theta_t + z\cos\theta_t)]}$$

Let's ignore the $e^{i\omega t}$ term,

$$E_\perp^t \Big|_{\theta_i > \theta_c} = \hat{a}_y t_\perp E_0 \exp\left[-ik_2 x\left(\frac{n_1}{n_2}\sin\theta_i\right)\right] \exp\left(-ik_2 z\sqrt{1-\sin^2\theta_t}\right)$$

$$\Big\Downarrow \text{ convert } \theta_t \text{ into } \theta_i$$

$$E_\perp^t \Big|_{\theta_i > \theta_c} = \hat{a}_y t_\perp E_0 \exp\left(-k_2 z\sqrt{\frac{n_1^2}{n_2^2}\sin\theta_i^2 - 1}\right) \exp\left[-i k_2 x\left(\frac{n_1}{n_2}\sin\theta_i\right)\right]$$

$$\boxed{= \hat{a}_y t_\perp E_0 e^{-\alpha z} e^{-i\beta x}}$$

where $\alpha = k_2\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}\Big|_{\theta_i > \theta_c}$ (real)

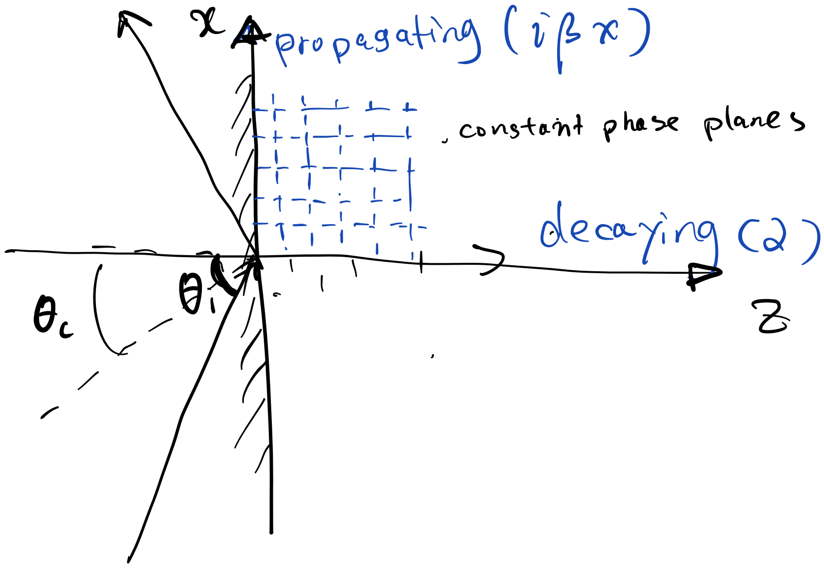
$$\beta = k_2 \frac{n_1}{n_2}\sin\theta_i \Big|_{\theta_i > \theta_c}.$$

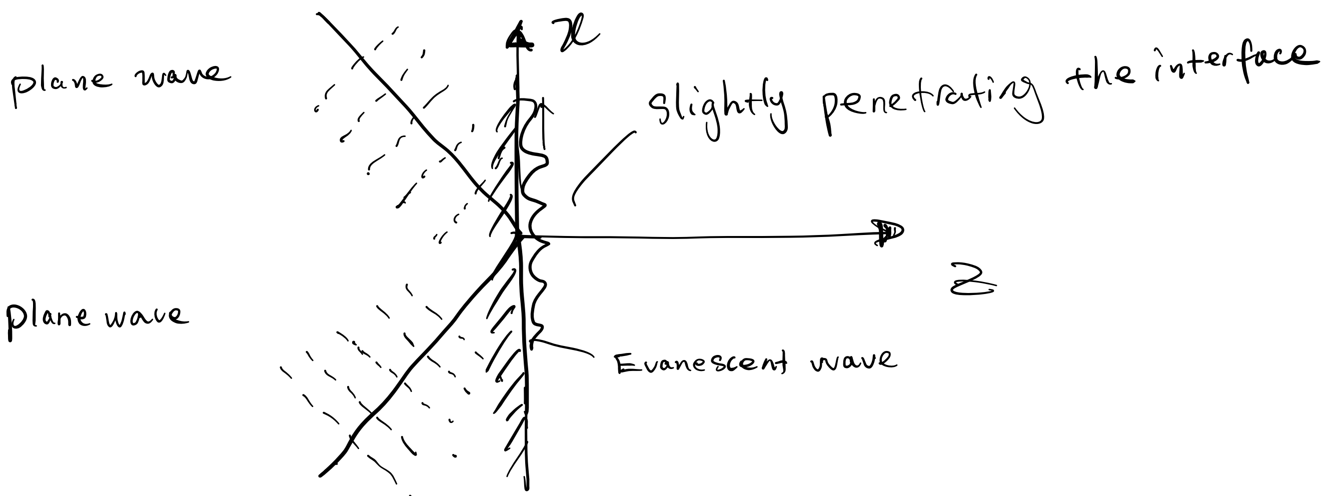
phase velocity of this wave

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{k_2 \frac{n_1}{n_2}\sin\theta_i} < \frac{\omega}{k_2} = v_{p2}$$

Physical meaning

① when $\theta_i > \theta_c$, $E_\perp^t$ is a surface wave, with amplitude decaying rapidly in a direction normal to the interface. (tightly bound to the surface)





Show animation!

② $\underline{v_p < v_{p2}}$. tightly bounded slow surface wave

Summary:

① when $\theta_i < \theta_c$, real power is transmitted into medium 2

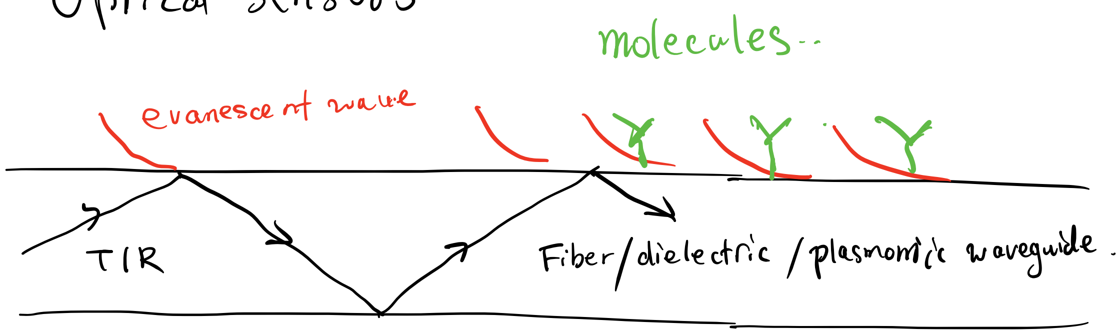
② when $\theta_i = \theta_c$, $\theta_t = 90°$, E^t is a surface wave along $\hat{x}$ direction (parallel to the interface), No real power transfer to the medium 2. (All reflected)

③ when $\theta_i > \theta_c$, there's a surface wave along $\hat{x}$, and is heavily attenuated along $\hat{z}$, no real power transferred into medium 2 (all reflected)

this wave exists because the boundary condition on the continuity of tangential component of $\vec{E}$ and $\vec{H}$

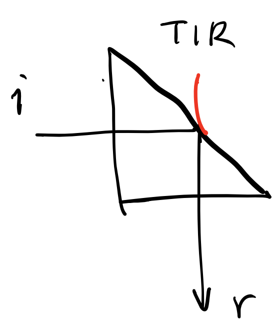
Application of evanescent wave

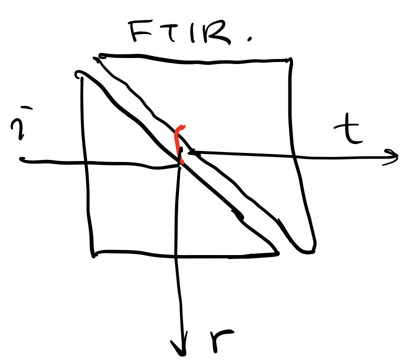
eg. 1 Optical sensors



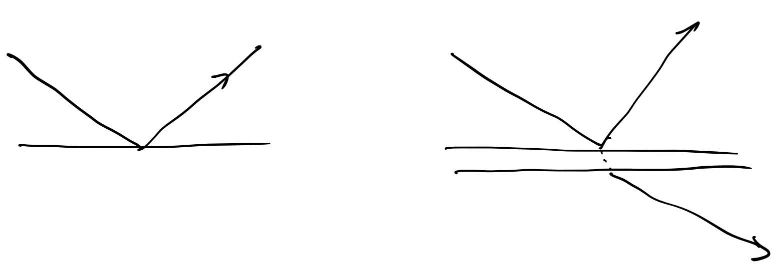
The absorbance of chemical/bio/gas molecules will interact the evenescent field, and perturb the transmitted optical signal.

eg. 2. Frustrated total interal reflection (FTIR)





Can be used to make beam splitters.



Homework 1.