Lecture 2: Plane Wave

Learning Objectives:

1. Wave equation in vaccum
2. why light is E&M wave?
3. properties of E&M wave?
   (energy and momentum)
4. Wave equations in linear and nonlinear media.

1. Wave equation in vacuum

In vacuum, there is no source, i.e. $J=0$, $\rho=0$

$$\begin{cases} \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & ① \\ \nabla\times\vec{B} = \mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t} & ② \end{cases}$$

$\nabla\times ①$:

$$\nabla\times(\nabla\times\vec{E}) + \frac{\partial}{\partial t}(\nabla\times\vec{B}) = 0$$

$$\xrightarrow{\text{plug in ②}} \nabla\times(\nabla\times\vec{E}) + \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} = 0$$

use $\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot E) - \nabla^2 E$ and $\nabla\cdot\vec{E} = 0$

$$\Rightarrow \quad \nabla^2\vec{E} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = 0 \quad ③$$

Similarly, $\nabla^2\vec{B} - \mu_0\epsilon_0\dfrac{\partial^2}{\partial t^2}\vec{B} = 0$ ④

Recall that

1D wave equation: $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{v^2}\dfrac{\partial^2 u}{\partial t^2} = 0$

3D wave equation: $\nabla^2 u - \dfrac{1}{v^2}\dfrac{\partial^2 u}{\partial t^2} = 0$

(wate wave, air wave, earth wave)

Comments:

1. ③ and ④ are wave equations if

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \equiv c \quad (\text{speed of light in vacuum})$$

speed of light is determined by eletro/magnet-statics.

(ROSA / DORSEY - 1907)

$c = 299,788$ km/s.

2. Light is an electromagnetic wave!!

3. ε_0, μ_0 are measured quantities by using charged pith balls, batteries and wires, having nothing to do with light.

4. Without Maxwell's contribution on displacement current, i.e. $J_d = \varepsilon_0 \frac{d\vec{E}}{dt}$, the wave eq. would not emerge, and there would be no electromagnetic theory of light!

5. Any superposition (linear sum) of solutions is a solution of wave equation.

2. What is wave?

Something varies both in space and time.

$y = \cos(Ax)$ not a wave!

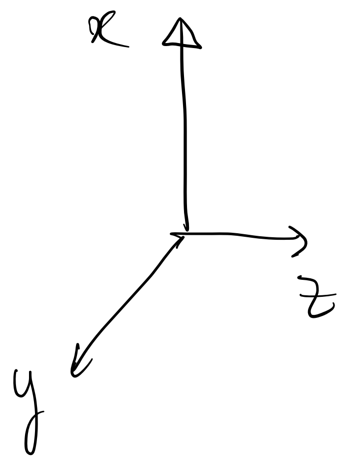
There are many types of solutions of wave equations.

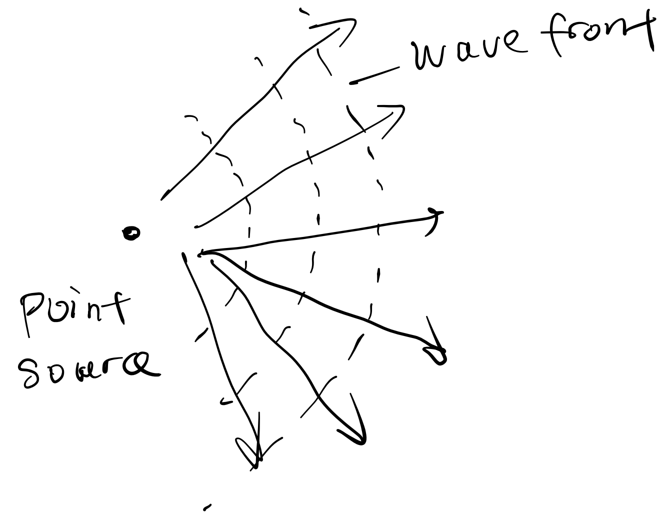
This is a solution

$E(z,t) = 5z - 3t$

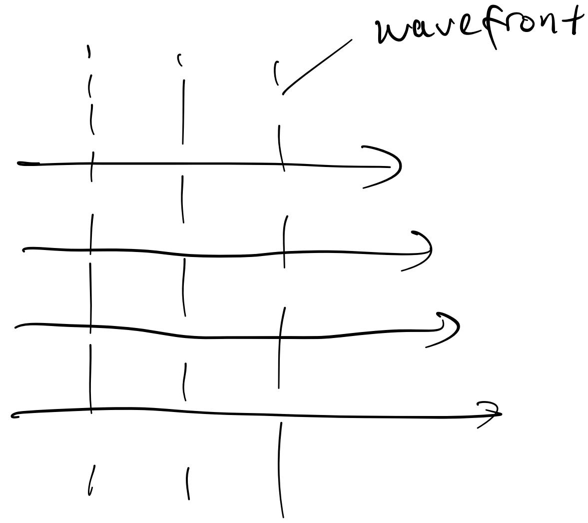
(but it is not a useful solution!!)

Spherical wave





plane wave



3. Monochromatic plane wave

There are many solutions of wave equation.

One possible/simplest solution is plane wave

$$\boxed{\begin{aligned} \vec{E}(x,y,z,t) &= \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{B}(x,y,z,t) &= \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \end{aligned}}$$

To satisfy wave equation,

$$\vec{k} = \omega\sqrt{\epsilon_0\mu_0}\,\hat{k} = \frac{\omega}{c}\hat{k},$$ is wavevector,

which denotes wave propagation direction,

$$|\vec{k}| = \frac{2\pi}{\lambda}$$

This solution is only valid when

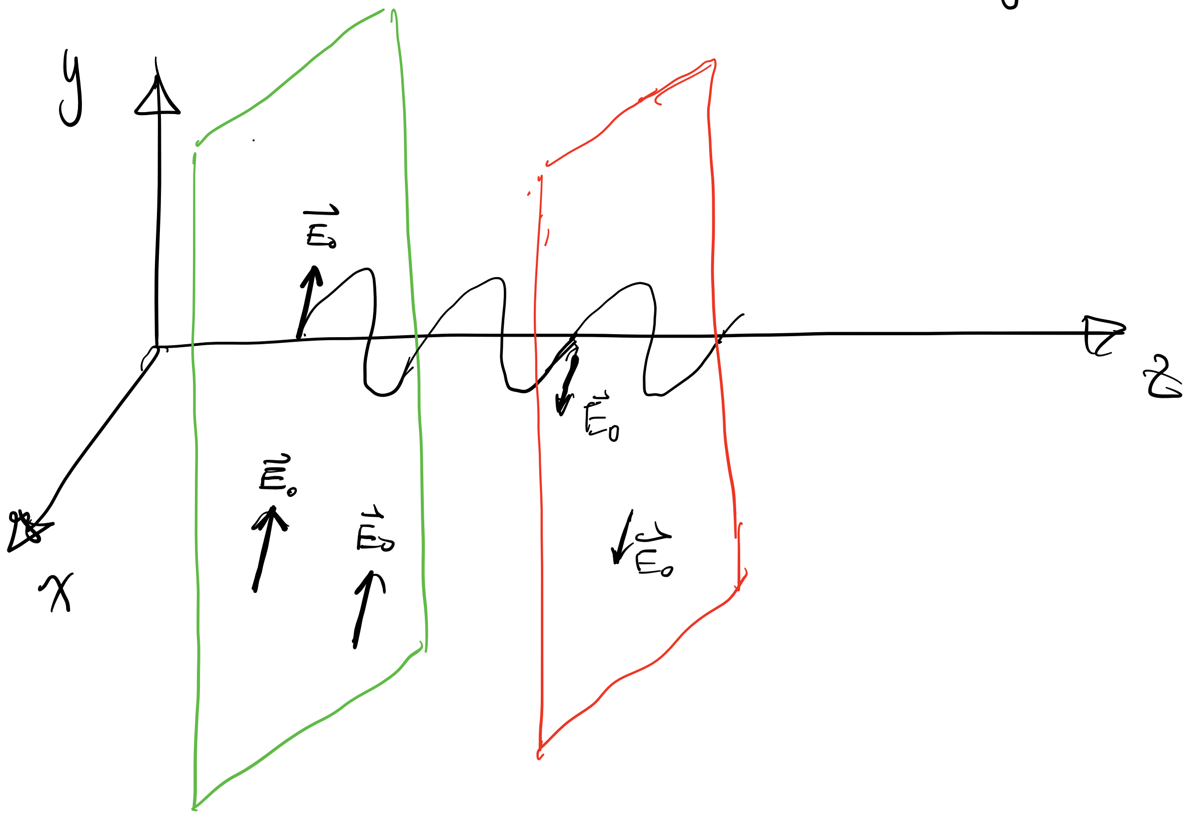
① ϵ, μ does not vary in space (isotropic)

② No free (moving) charge

③ No current (J)

Illustration of plane wave:

Let's assume the wave is propagating in $\hat{z}$

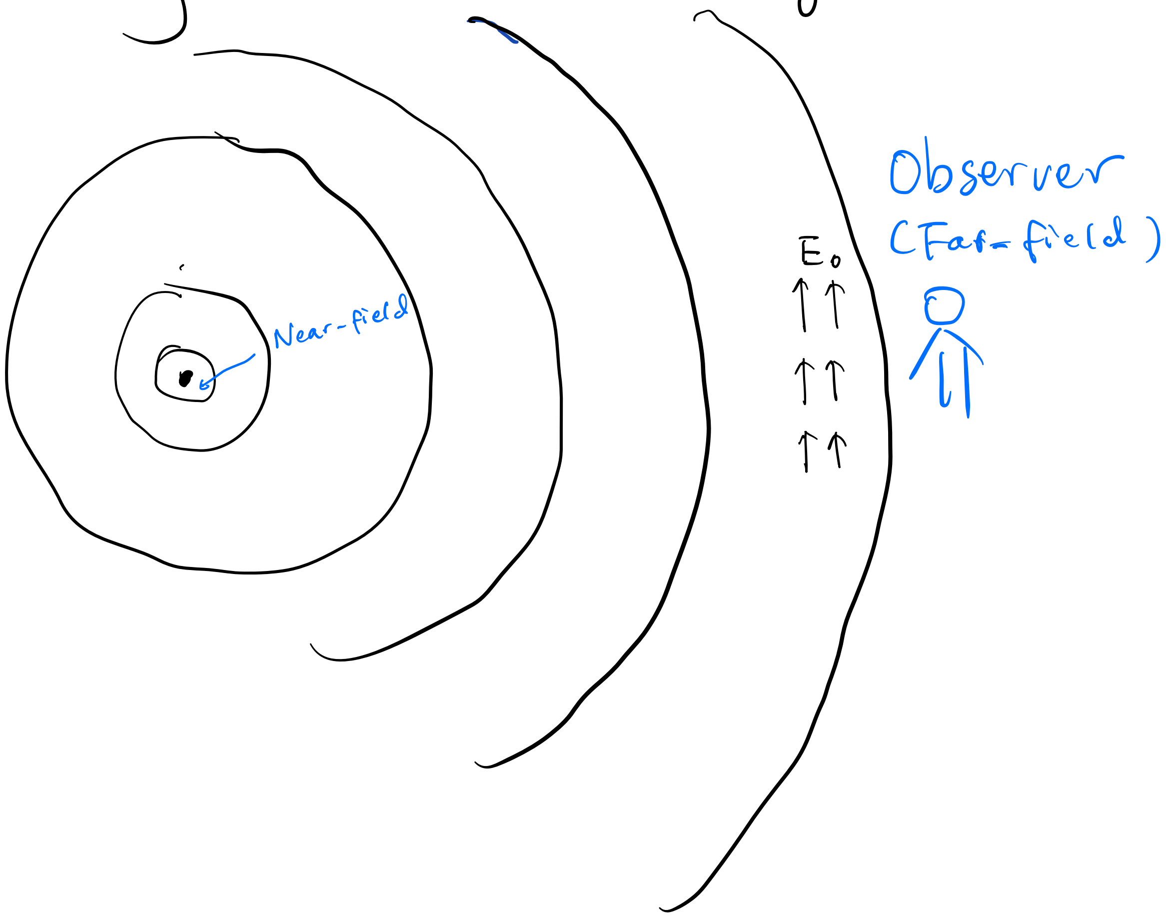


① In the plane orthogonal to the propagation direction the vector field $\vec{E}_0$ is uniform (same direction, same amp.)

② At different planes, $\vec{E}_0$ might be different.

Why plane waves are useful solutions?

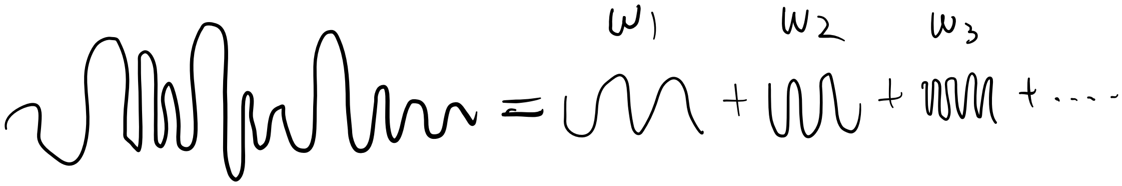
Assuming a point source emitting E&M radiations.



Plane waves are good approximations when you are far from the source of E&M radiation (far field)

In most cases, we are dealing with far-field, for near-field optics, plane wave is not a good Appr.

Arbiturary wave



Nearly any waveform in space or time can be made up by summing enough plane waves.

(Fourier Theory!)

4. Electromagnetic plane waves in vaccum.

What are the relations between $\vec{E}$ and $\vec{B}$?

Consider $\vec{E} \sim e^{i(\vec{k}\vec{r} - \omega t)}$

① Spatial derivitives $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

$$\nabla\left(e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right) = i\vec{k}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\boxed{\nabla \to i\vec{k}}$$

② Time derivatives

$$\frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r} - \omega t)} = -i\omega e^{i(k\omega - \omega t)}$$

$$\boxed{\frac{\partial}{\partial t} = -i\omega}$$

Although $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$, $\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$ are solutions of wave eq, they might not satisfy Maxwell eqs.

① $\nabla \cdot \vec{E} = 0$, $\nabla \cdot \vec{B} = 0$ requires

$$i\vec{k} \cdot \vec{E} = 0, \quad i\vec{k} \cdot \vec{B} = 0$$

physical meaning: both $\vec{E}$ and $\vec{B}$ are $\perp$ to $\vec{k}$

② $$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} \rightarrow i\vec{k} \times \vec{E} = i\mu_0 \omega \vec{H}$$

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \rightarrow i\vec{k} \times \vec{H} = -i\epsilon_0 \omega \vec{E}$$

physical meaning: $\vec{E}$, $\vec{H}$, $\vec{k}$ are mutally orthogonal.

$$\vec{E} = \hat{x} E_0 e^{i(kz-\omega t)}$$

$$\hat{H} = \hat{y} H_0 e^{i(kz-\omega t)}$$

plug in (2), we have

$$i(\underbrace{\vec{k} \times \hat{x}}_{k\hat{y}}) E_0 e^{i(kz-\omega t)} = i\hat{y}[\mu_0 \omega H_0 e^{i(kz-\omega t)}]$$

$$\Rightarrow i k \hat{y} E_0 \cancel{e^{i(kz-\omega t)}} = i\hat{y}[\mu_0 \omega H_0 \cancel{e^{i(kz-\omega t)}}]$$

$$\boxed{E_0 = \mu_0 \frac{\omega}{k} H_0 = \mu_0 c H_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} H_0}$$

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv Z_0 = 377\ \Omega$$

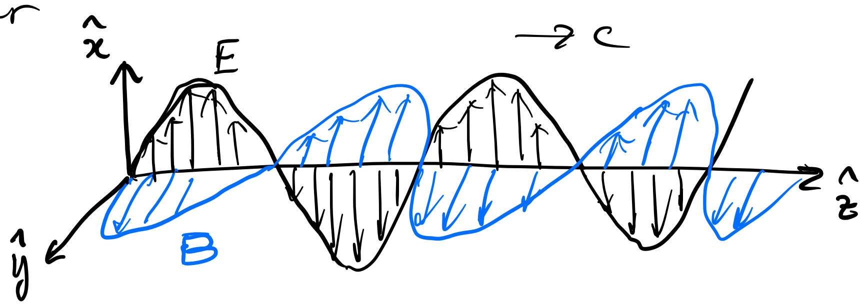
(impedance of free space)

Or $\boxed{B = \mu_0 H = \frac{E_0}{c}}$

Comments:

① E & M in vaccum is transverse wave (TEM wave)

② $\tilde{E}$ and $\tilde{B}$ are in phase and mutally perpendicular



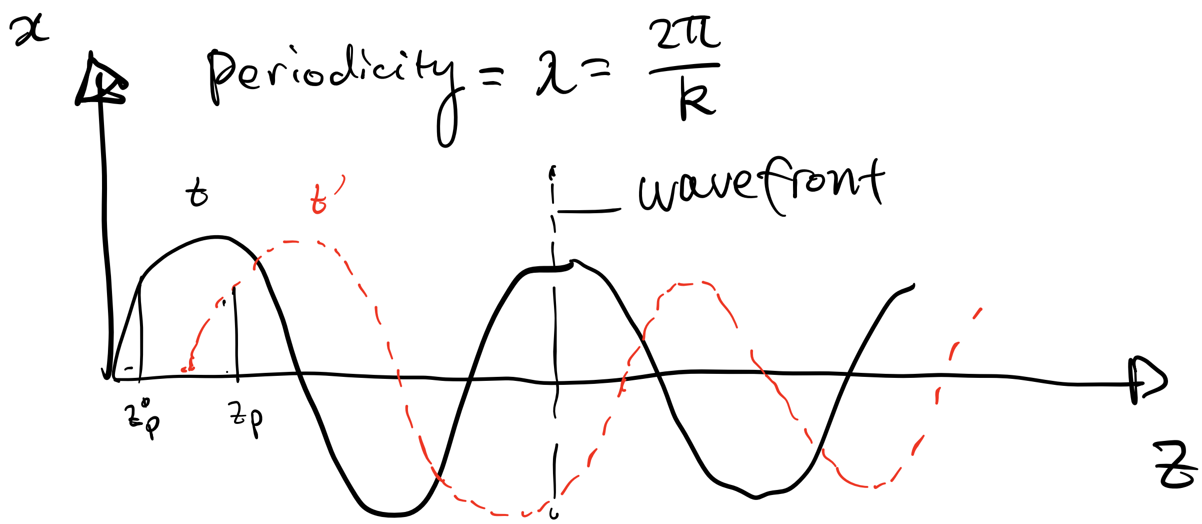
③ Amplitude of magnetic field is very small compare to $\vec{E}$-field, we can ignore the contribution of $\vec{B}$ in photonics

④ The direction of $\vec{E}$ is called polarizatio

5. Phase, wavevector, wavelength

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)} = E_0 e^{i\phi}$$

Periodicity $= \lambda = \frac{2\pi}{k}$



$\phi = kz - \omega t = c$

$z = \frac{c + \omega t}{k}$, $\boxed{v_p = \frac{dz}{dt} = \frac{\omega}{k}}$

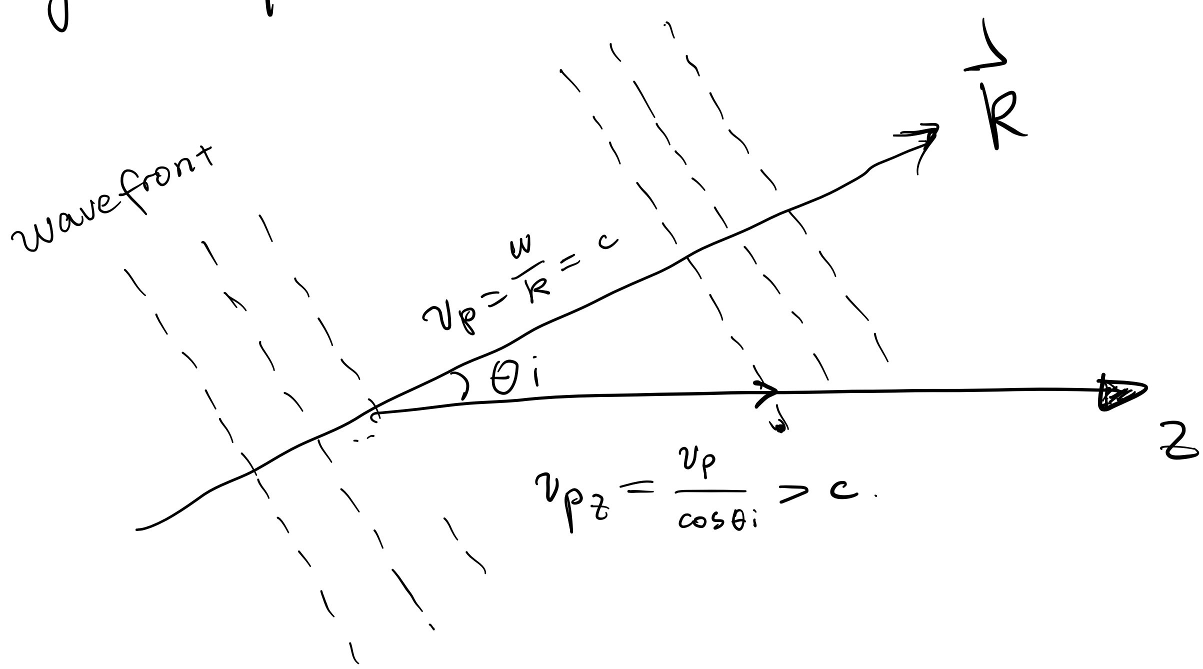
Phase velocity: velocity of wavefront.
(or constant phase planes)

Comments:

① The phase velocity is a hypothetic speed

② The phase velocity can exceed c

e.g. Oblique incidence



6. Enery, momentum of E&M waves

① energy per unit volume of E&M field

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

where $B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2$

Electric and magnetic fields have equal contributions.

$$u = \epsilon_0 E^2 = \epsilon_0 E^2 \cos^2(kz - \omega t + \delta)$$

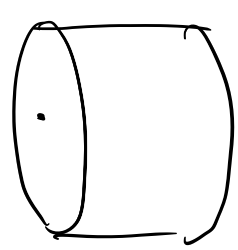
② Enery flux density (energy per unit area, per unit time) transported by the field

$$S = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = c\,\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z}$$

$$= c u \hat{z}$$

<u>Meaning</u>:

A



$c \cdot \Delta t$

total energy =

$u \cdot \Delta V = u A c \Delta t$

$= S \cdot A \cdot \Delta t$

EM field also carries momentum

moment density stored in the field is:

$$\vec{g} = \frac{1}{c^2}\vec{S}$$

For monochromatic plane waves:

$$\hat{g} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z} = \frac{1}{c}u\hat{z} = \frac{\epsilon_0 E_0^2}{c}\cos^2(kz - \omega t + \delta)$$

For light, wavelength is so short, any macroscopic measurement will encompass many cycles.

Note, the average of cosine-squared over a complete cycle is $1/2$, so,

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$\langle \vec{S} \rangle = \frac{1}{2}c\epsilon_0 E_0^2$$

$$\langle \vec{g} \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \hat{z}$$

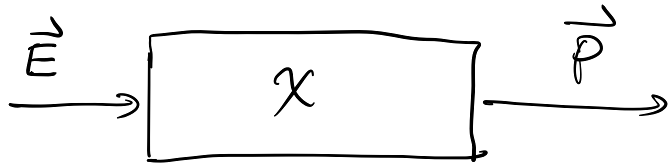
$\langle\ \rangle$ denotes time-average over a complete cycle.

The average power per unit area transported by an E&M wave is called <u>intensity</u>

$$\boxed{I = \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2}$$

6. Electromagnetic waves in dielectric media.

① Linear and Homogeneous media



χ is uniform in space

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

$$\begin{cases} \nabla \times \vec{H} = \epsilon_0 \epsilon_r \dfrac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{E} = -\mu_r \mu_0 \dfrac{\partial \vec{H}}{\partial t} \\ \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{H} = 0 \end{cases}$$

identical to free space Maxwell equation, but just replace μ_0, ϵ_0 by $\epsilon_0 \epsilon_r$ and $\mu_0 \mu_r$.

So the wave eq. in Linear and Homogeneous media is

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

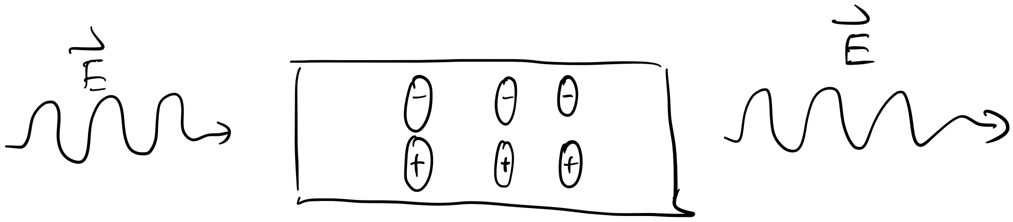
$$c = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_r \mu_0}} = \frac{c_0}{\sqrt{\epsilon_r}}, \text{ or } \frac{c_0}{n}$$

c_0: vacuum light speed.

n: refractive index

Comments:

① It looks as if light is "slowed down" by materias

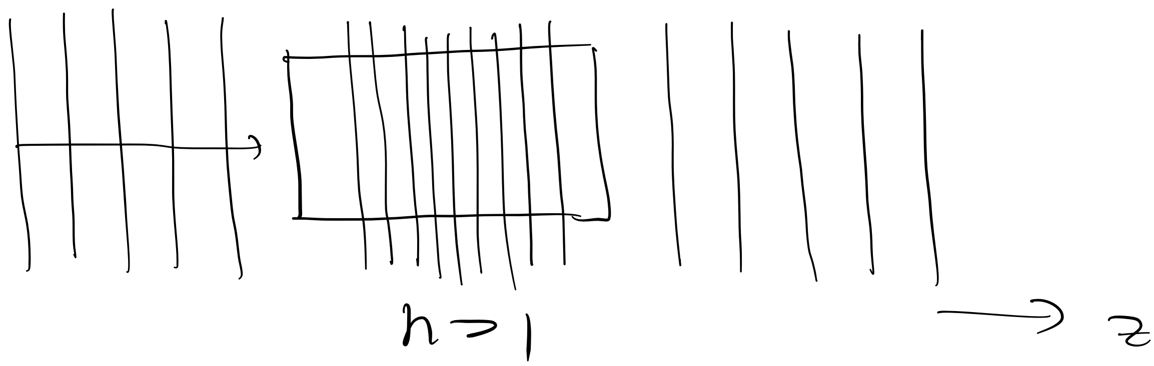


<u>Reason</u>: it takes times for dipole to emit light again, will come back to this in Lecture 4.

wave vector $k = \frac{\omega}{c} = n\frac{\omega}{c_0}$

$$\lambda = \frac{2\pi}{k} = \frac{\lambda_0}{n}$$

wavefronts.



② Wavelength is shorter in materials.

② Nonlinear and Homogeneous media

$$P = \epsilon_0 \chi^{(1)} E + \epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} E^3 + \cdots$$

$$D = \epsilon_0 E + P = \epsilon_0 E + \underbrace{\epsilon_0 \chi^{(1)} E}_{P_L} + \underbrace{\epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} E^3 + \cdots}_{P_{NL}}$$

In this case, we can no longer define a ϵ_r such that $D = \epsilon_0 \epsilon_r E$, we have to write $\vec{D} = \epsilon_0 E + \vec{P}$.

Then we have a generalized wave eq.

$$\nabla^2 E - \frac{1}{c_0^2} \frac{\partial E^2}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2 E - \frac{1}{c_0^2} \frac{\partial E^2}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}}$$

For linear optics, $P_{NL} = 0$,

For nonlinear optics, $P_{NL} \neq 0$,