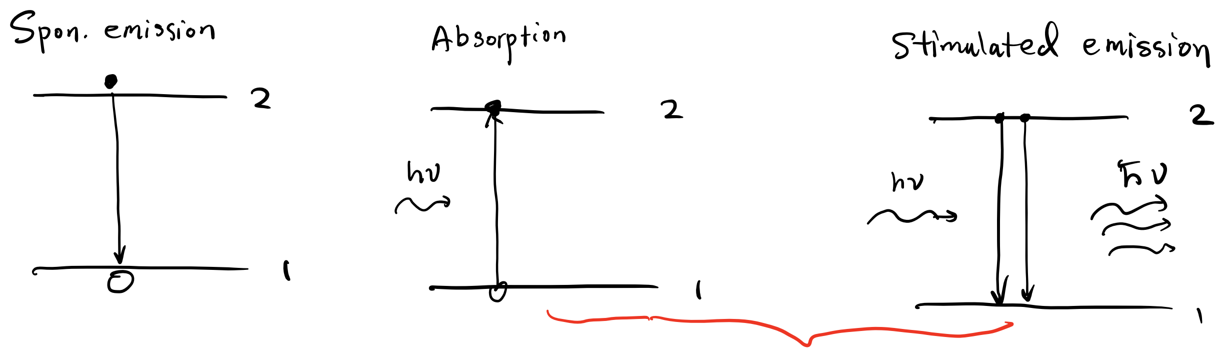
# Lecture 14. Stimulated emission and amplification

Learning objectives

1. Spontaneous v.s. stimulated emission.
2. Population inversion, amplification and gain
3. gain saturation $\begin{cases} \text{homogeneous} \\ \text{inhomogenous.} \end{cases}$

## 1. Spontaneous emission v.s. Stimulated emission ($P_{225}$ Yariv)

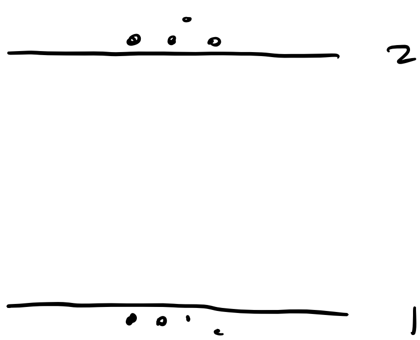



Similarity: Induced transitions!

Difference between spontaneous emission and stimulated emission?

> rate for stimulated emission $\sim$ intensity of incident $\vec{E}$
>
> 〃 〃 spontaneous emission is independent of incident $\vec{E}$

Problem to solve: Stimulate emission rate as a function of $\vec{E}$




Downward transition rate $(2 \to 1)$

$$W_{21} = \underbrace{B_{21}\,\rho(\nu)}_{\text{Stimulated}} + \underbrace{A_{21}}_{\text{spon. emission}}$$

Upward transition rate $(1 \to 2)$

$$W_{12} = \underbrace{B_{12} \cdot \rho(\nu)}_{\text{Absorption}}$$

**Note:** $B_{21}$, $B_{12}$ are called spon. emission and absorption constant.

$\rho(\nu)$ is energy density per unit frequency.

At thermal equilibrium, the average populations of level 2 and 1 are constant with time, i.e.

\# of $2 \to 1$ transitions = \# of $1 \to 2$ transitions in a given time interval.

So $$N_2 \cdot W_{21} = N_1 \cdot W_{12}$$

equilibrium population density (#/m³)

$$\Rightarrow N_2 \left( B_{21} \rho(\nu) + A_{21} \right) = N_1 B_{12} \rho(\nu)$$

index of host medium

At temperature T, $\rho(\nu) = \frac{8\pi n^3 h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$

$$\Rightarrow N_2 \left( B_{21} \frac{8\pi n^3 h\nu^3}{c^3 \left(e^{\frac{h\nu}{kT}} - 1\right)} + A_{21} \right) = N_1 \left( B_{12} \frac{8\pi n^3 h\nu^3}{c^3 \left(e^{\frac{h\nu}{kT}} - 1\right)} \right)$$

and also in thermal equilibrium,

$$\frac{N_2}{N_1} = e^{-\frac{h\nu}{kT}} \quad \text{(Boltzman factor)}$$

$$\Rightarrow \frac{8\pi n^3 h\nu^3}{c^3 \left(e^{\frac{h\nu}{kT}} - 1\right)} = \frac{A_{21}}{B_{12} e^{\frac{h\nu}{kT}} - B_{21}} \qquad (1)$$

To satisfy ①, we must have

$$\begin{cases} B_{12} = B_{21} \\ \dfrac{A_{21}}{B_{21}} = \dfrac{8\pi n^3 h\nu^3}{c^3} \end{cases}$$

So stimulated emission rate

$$W_i = B_{21} \cdot \rho(\nu) = \frac{A_{21} c^3}{8\pi n^3 h\nu^3} \rho(\nu) \overset{t_{spon} = \frac{1}{A_{21}}}{=\!=\!=} \frac{c^3}{8\pi n^3 h\nu^3 t_{spon}} \rho(\nu)$$

where $t_{spon} = 1/A_{21}$ is the spontaneous lifetime of atom.

Consider the brodening of the emission spectrum,

$$W_i = \int \frac{c^3}{8\pi n^3 h\nu^3 t_{spon}} \rho(\nu) g(\nu) d\nu$$

Let $U_0$ be the energy density (J/V) of the monochromatic field, i.e. $\rho(\nu') = U_0 \delta(\nu' - \nu)$

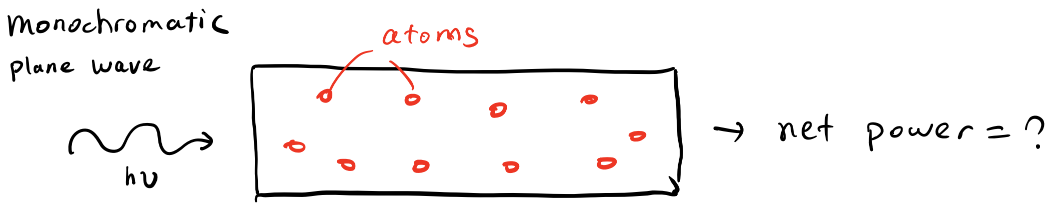
$$\Rightarrow W_i = \frac{c^3 U_0}{8\pi n^3 h\nu^3 t_{spon}} g(\nu) \overset{I_0 = \frac{cU_0}{n}}{=\!=\!=} \frac{\lambda^2 I_0}{8\pi n^2 h\nu t_{spon}} g(\nu)$$

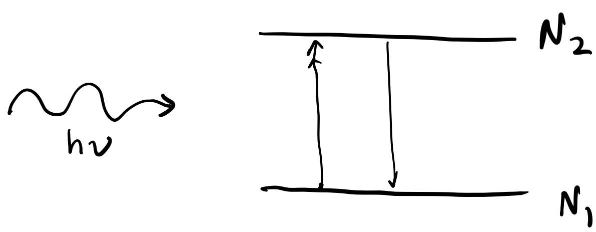
(induced transition rate due to the presence of a monochromatic field)

where $I_0$ is the beam intensity ($W/m^2$)

<u>Conclusion:</u> induced transition rate ~ intensity of incoming optical beam.

## 2. Absorption and amplification.




$N_2$ (# of $e^-$ per unit volume)

### Induced transitions:

$2 \to 1$ : $N_2 . W_i$ per unit time per unit volume

$1 \to 2$ : $N_1 W_i$ per unit time per unit volume.

Net power generated pe unit volume :

$$\frac{P}{\text{Volume}} = (N_2 - N_1) W_i h\nu$$

$$= (N_2 - N_1) \frac{\lambda^2 I_0}{8\pi n^2 \cancel{h\nu}\, t_{spon}} g(\nu) . \cancel{h\nu}$$

We know $I_0 = \dfrac{P}{A}$.

$$\frac{dI_0}{dz} = (N_2 - N_1) \frac{c^2 g(\nu)}{8\pi n^2 \nu^2 t_{spon}} I_0 \qquad (1)$$

Solution of ① is

$$\boxed{I_o(z) = I_o(0)\, e^{\gamma(\nu) z}}$$

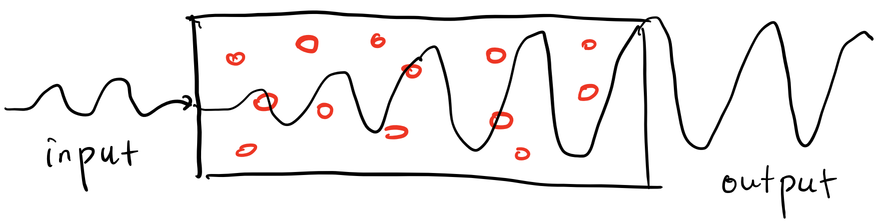
where $\gamma(\nu) = (N_1 - N_2) \dfrac{c^2}{8\pi n^2 \nu^2 t_{spon}} g(\nu)$ is the gain coefficient.

## Comments:

① When $N_1 > N_2$ (population inverted), $I_o$ grows exponentially.

This corresponds to laser-type amplification at population inversion.

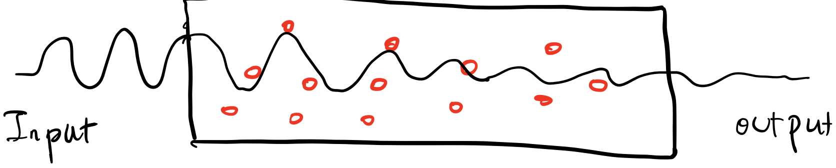
Amplifying medium $(N_2 > N_1)$




② When $N_1 < N_2$, $I_o$ attenuates.

This corresponds to absorption

Absorbing medium $(N_2 < N_1)$



③ At thermal equilibrium. $\frac{N_2}{N_1} = e^{-\frac{h\nu}{kT}}$, $(N_2 < N_1)$

Density of absorbed power: (homework 2 problem)

$$\frac{P_{abs}}{Volume} = \frac{1}{2}\omega\epsilon_0 \underbrace{\chi''(\nu)}_{\text{img. part of the electric susceptibility}} |E|^2$$

$$= (N_1 - N_2) W_i h\nu$$

$$\Rightarrow \chi''(\nu) = \frac{(N_2 - N_1)\lambda^3}{16\pi^2 \cdot n \cdot t_{spon}} g(\nu).$$

here, we used $I_0 = \frac{c}{n}\epsilon|E|^2/2$

For Lorentzian lineshape of $g(\nu)$,

$$\boxed{\chi''(\nu) = \frac{(N_2 - N_1)\lambda^3}{8\pi^3 n\, t_{spon}\,\Delta\nu} \frac{1}{1 + [4(\nu - \nu_0)^2]/(\Delta\nu)^2}}$$

very important !!

also $\gamma(\nu) = -\frac{2\pi}{\lambda n}\chi'' = -\frac{k}{n^2}\chi''$

For passive media, $\chi'' > 0$. $\gamma(\nu) < 0$, (no gain)

gain coefficient of an active medium with Lorentzian lineshape

$$\gamma(\nu) = \frac{(N_2 - N_1)\lambda^2}{4\pi^2 n^2 t_{spon}\,\Delta\nu} \frac{1}{1 + \left[4(\nu-\nu_0)^2\right]/(\Delta\nu)^2}$$

gain at the line center:

$$\gamma_0 = \gamma(\nu_0) = \frac{(N_2 - N_1)\lambda^2}{4\pi^2 n^2\, t_{spon}\,\Delta\nu}.$$

So $$\gamma(\nu) = \frac{\gamma_0}{1 + \left[4(\nu-\nu_0)^2\right]/(\Delta\nu)^2}$$

Complex refractive index of the laser medium

$$n'^2 = \underbrace{n^2}_{\text{index of the host medium}} + \underbrace{\chi' - i\chi''}_{\text{dielectric susceptibility of active atoms}}$$

Complex wave vector of propagation in laser medium

$$k' = k_o n' = k_o (n^2 + \chi' - i\chi'')^{1/2} \simeq n k_o \left(1 + \frac{\chi'}{2n^2} - i\frac{\chi''}{2n^2}\right)$$

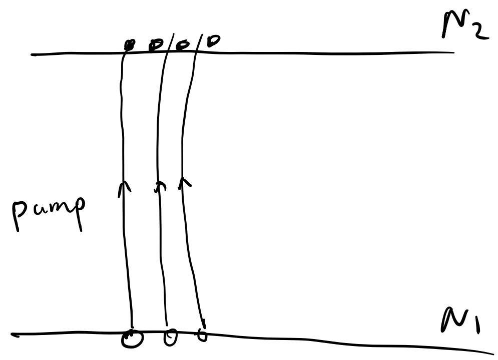
Let $k = n.k_o$ (wave vector in host material)

$$k' = k\left(1 + \frac{\chi'}{2n^2} - i\frac{\chi''}{2n^2}\right) = k + k\frac{\chi'}{2n^2} - ik\frac{\chi''}{2n^2}$$

# 3. Gain saturation in homogeneous laser media

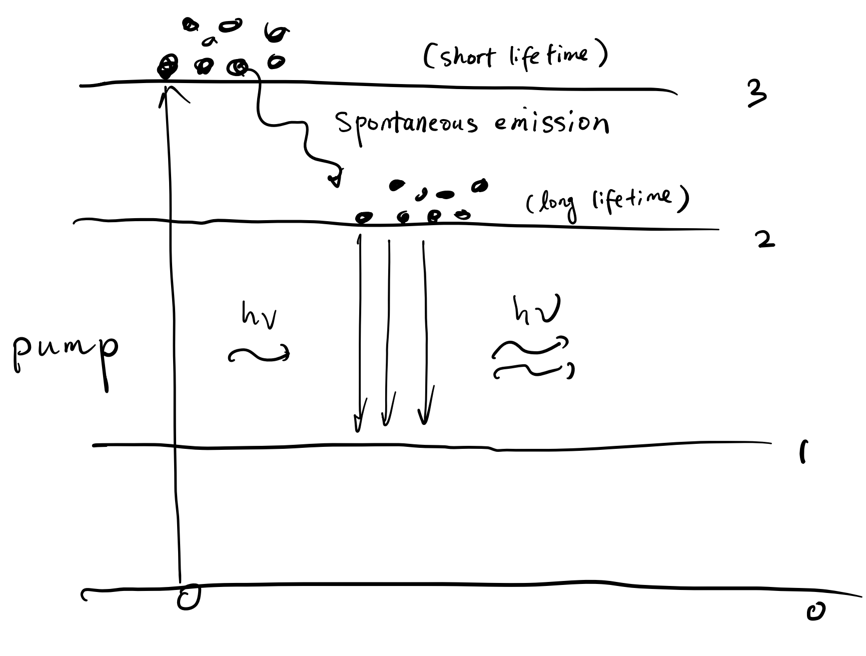
How to creat gain ($N_2 - N_1 > 0$) ?

2-level system




$N_2$ is always less than $N_1$

Four-level system



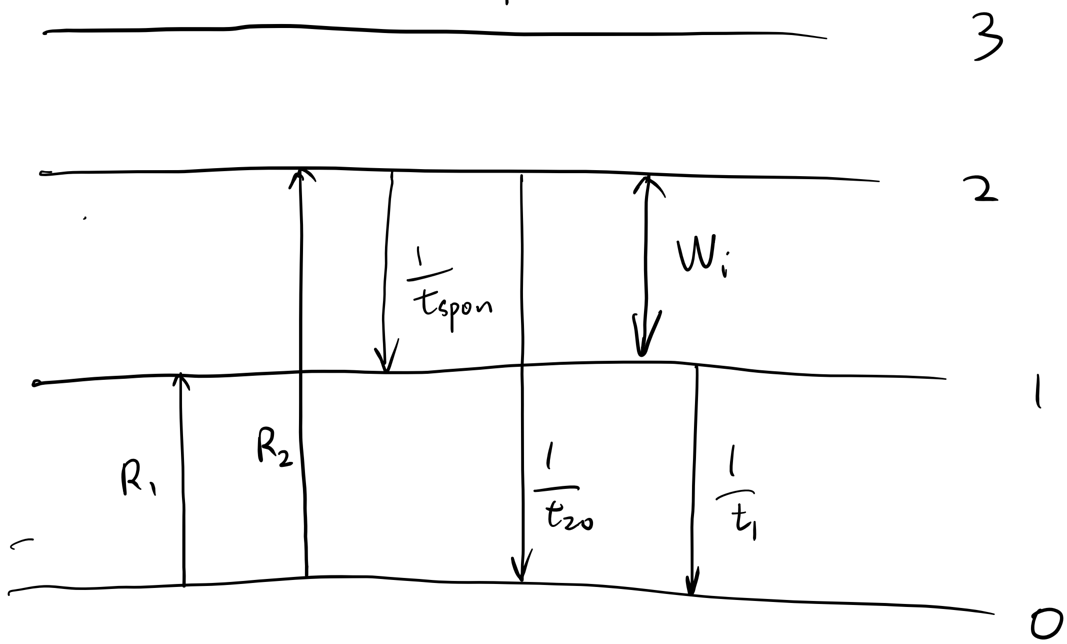

$N_2 > N_1$

Problem to study:

Q: how does $N_2 - N_1$ (or gain) change when there is eletromagnetic field?

A: Solve the rate equations for level 2 and 1 in steady state!

Four-level system




Rate equations:

$$\begin{cases} \dfrac{dN_2}{dt} = R_2 - \dfrac{N_2}{t_2} - (N_2 - N_1) W_i & \left( \dfrac{1}{t_2} = \dfrac{1}{t_{spon}} + \dfrac{1}{t_{20}} \right) \\ \dfrac{dN_1}{dt} = R_1 - \dfrac{N_1}{t_1} + \dfrac{N_2}{t_{spon}} + (N_2 - N_1) W_i & \end{cases}$$

Where $N_1, N_2$ are population densities ($m^{-3}$) of level 2 and 1.

$R_2, R_1$ are pumping rate ($m^{-3} s^{-1}$)

Assumptions: all atoms are same (homogeneous!!)

Steady state: $\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0$,

$$N_2 - N_1 = \frac{R_2 t_2 - \left(R_1 + \frac{t_2}{t_{spon}}\right) t_1}{1 + \left[t_2 + \left(1 - \frac{t_2}{t_{spon}}\right) t_1\right] W_i(\nu)}$$

If no optical field, $W_i(\nu) = 0$,

$$\Delta N^0 = R_2 t_2 - (R_1 + \delta R_2) t_1$$

population inversion without field

So $$N_2 - N_1 = \frac{\Delta N^0}{1 + \phi\, t_{spon} W_i(\nu)}$$

where $$\phi = \frac{t_2}{t_{spon}} \left(1 + \left(1 - \frac{t_2}{t_{spon}}\right) \frac{t_1}{t_2}\right)$$

Note: $t_2 < t_{spon}$, for efficient lasers, $t_2 \simeq t_{spon}$, $t_1 \ll t_2$,

So $\phi \simeq 1$

So $$N_2 - N_1 = \frac{\Delta N_0}{1 + \left[\frac{\phi \lambda^2 g(\nu)}{8\pi n^2 h\nu}\right] I_0} = \frac{\Delta N_0}{1 + \frac{I_0}{I_s(\nu)}}$$

where $I_s(\nu)$ is called "saturation intensity"

$$I_s(\nu) = \frac{8\pi n^2 h\nu}{\left(\frac{t_2}{t_{spon}}\right)\lambda^2 g(\nu)}$$

around the peak of the lineshape function $g(\nu)$

$$1/g(\nu) \simeq \Delta\nu .$$

$$\boxed{I_s(\nu) = \frac{8\pi n^2 h\nu\,\Delta\nu}{\left(t_2/t_{spon}\right)\lambda^2}}$$

Physical meaning?

when $I_0 = I_s$, population inversion drops to its half

$(N_2 - N_1 = \frac{1}{2}\Delta N^0)$

gain coefficient:

$$\gamma(\nu) = (N_2 - N_1)\frac{c^2}{8\pi n^2 \nu^2 t_{spon}} g(\nu)$$

$$= \frac{1}{1 + I_0/I_s(\nu)}\left(\frac{\Delta N^0 \lambda^2}{8\pi n^2 t_{spon}}\right) g(\nu) = \frac{\gamma_0(\nu)}{1 + I_0/I_s(\nu)}$$

where $\gamma_0(\nu)$ is the "small signal gain" when $I_0 \ll I_s$.