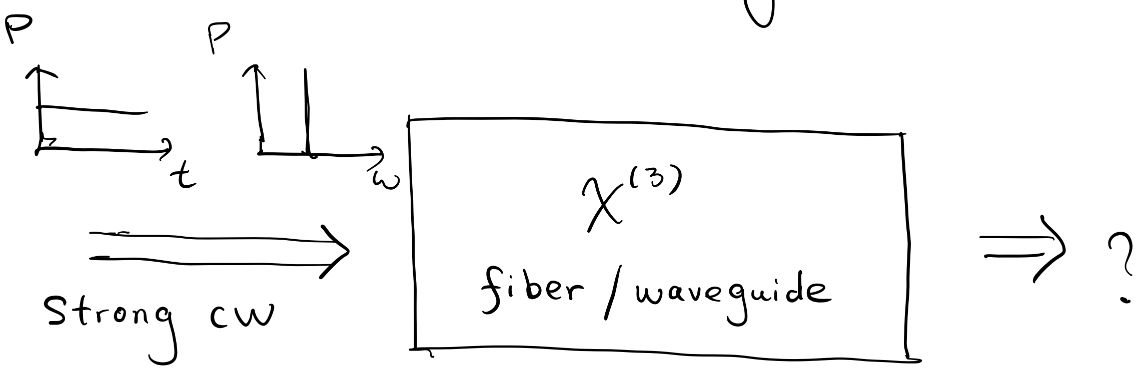
Lecture 11. Modulation instability, Soliton

Learning objectives:

1. Modulation instability (MI)
2. Optical solitons
3. Physical interpretation of soliton

1. Modulation Instability (MI)

Problem to study:



NLS:

$$\frac{\partial A(z,t)}{\partial z} + \underbrace{\frac{i\beta_2}{2}\frac{\partial^2 A(z,t)}{\partial T^2}}_{\text{Dispersion}} = \underbrace{i\gamma |A(z,t)|^2 A(z,t)}_{\text{Kerr-nonlinearity}} \qquad (1)$$

If input is CW ($\frac{\partial A}{\partial T}) = 0$, the steady-state solution is:

$$\bar{A} = \sqrt{P_0}\exp(i\phi_{NL})$$

$\phi_{NL} = \gamma \cdot P_0 z$

(nonlinear phase shift)

Power at $z=0$,

Physical meaning: CW light should propagate through the fiber unchanged except for acquaining a power-dependent phase shift!

Add a small perturbation:

$a(z,t)$, perturbation

$$A = (\sqrt{P_0} + a)\exp(i\phi_{NL})$$

Plug in (1), we get

$$\frac{\partial a}{\partial z} + i\frac{\beta_2}{2}\frac{\partial^2 a}{\partial T^2} = i\gamma P_0 (a + a^*)$$

Solution is in the form of

wavenumber, freq. of perturbation

$$a(z,T) = a_1 \exp[i(Kz - \Omega T)] + a_2 \exp[-i(Kz - \Omega T)]$$

where $$K = \pm \frac{1}{2}|\beta_2 \Omega| [\Omega^2 + \mathrm{sgn}(\beta_2)\Omega_c^2]^{1/2} \qquad (2)$$

$$\Omega_c^2 = \frac{4\gamma P_0}{|\beta_2|} = \frac{4}{|\beta_2| L_{NL}}$$

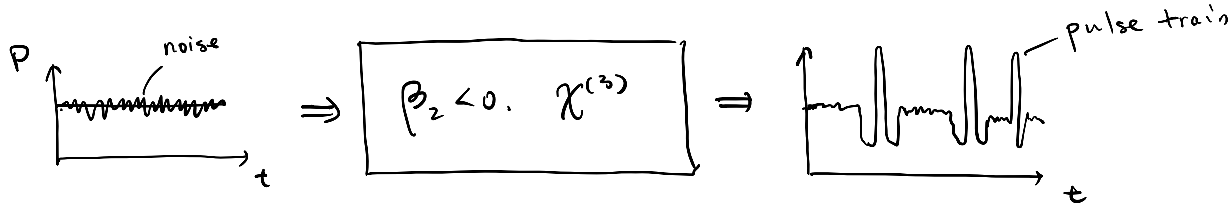
Comments:

(1) For normal GVD ($\beta_2 > 0$), K is always real,
Steady-state solution is stable against small perturbation!

(2) For anomalous GVD ($\beta_2 < 0$), K is imaginary when
$|\Omega| < \Omega_c$, perturbation $a(z,T)$ grows exponentially
with z — Modulation Instability.

(3) MI leads to a spontaneous temporal modulation of
CW beam, break CW into a pulse train. (Self-pulsing)

Visualization:



Modulalioion instability gain:

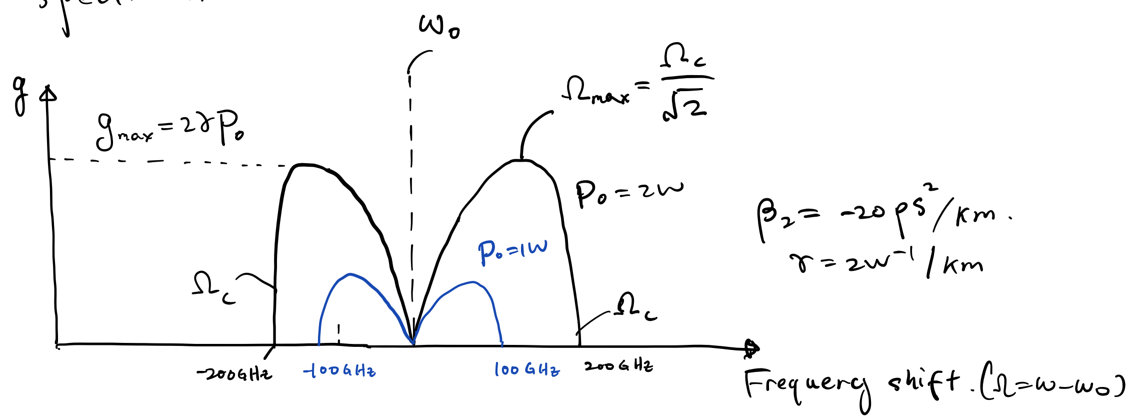
Set $\mathrm{sgn}(\beta_2) = -1$.

only occurs when $|\Omega| < |\Omega_c|$

Power gain: $g(\Omega) = 2\mathrm{Im}(K) = |\beta_2 \Omega| \sqrt{\Omega_c^2 - \Omega}$

$\Omega_c = \frac{4\gamma P_0}{|\beta_2|}$

gain spectrum:



$\beta_2 = -20\, ps^2/km$.
$\gamma = 2W^{-1}/km$

Maximum gain freq. $\Omega_{max} = \pm \frac{\Omega_c}{\sqrt{2}} = \pm \left(\frac{2\gamma P_0}{|\beta_2|}\right)^{1/2}$

Peak gain: $g_{max} = g(\Omega_{max}) = \frac{1}{2}|\beta_2|\Omega_c^2 = 2\gamma P_0$

Comments:

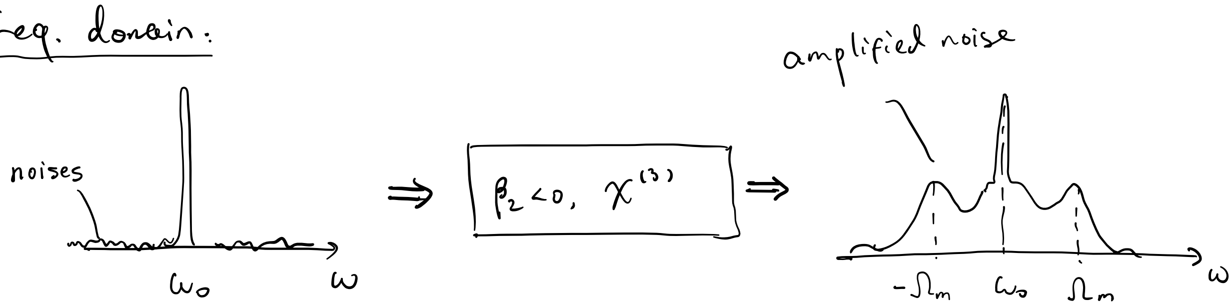
1. MI gain spectrum is symmetric with respect to $\Omega = 0$. at $\Omega = 0$, gain $= 0$

2. $P_0 \uparrow$. Maxium gain $\uparrow$. $\Omega_c \uparrow$. $\Omega_{max} \uparrow$

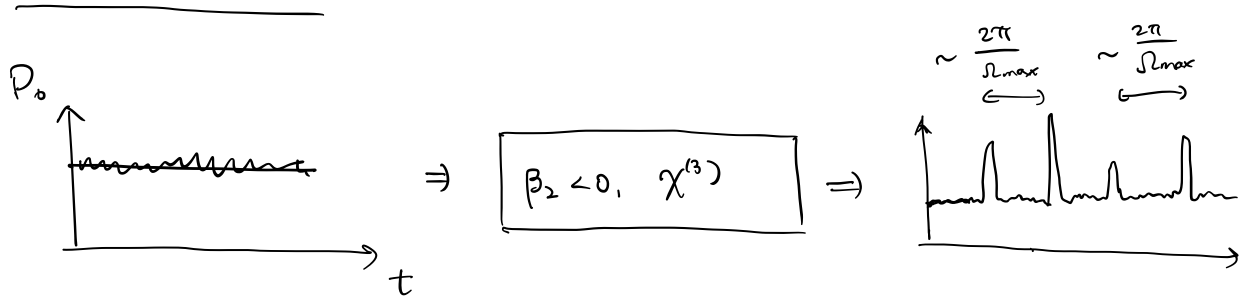
3. Detrimental for high-power fiber comm.

4. · In time domain, CW beam is converted into a periodic pulse train with a period. $T_m \sim \frac{2\pi}{\Omega_{max}}$.

Freq. domain:



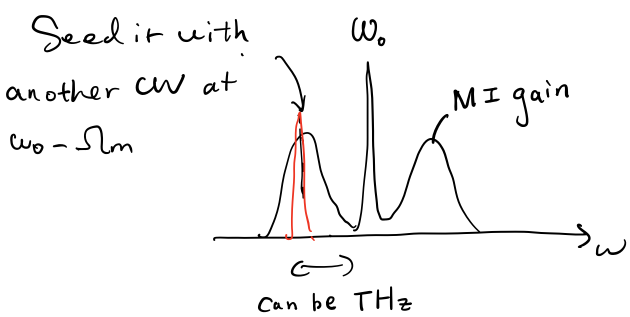
Time domain:

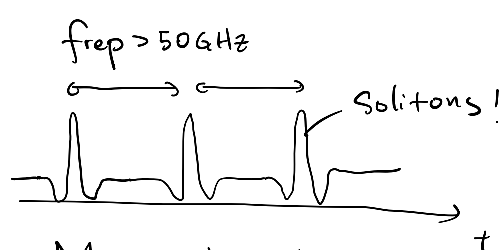


Applications:

① Supercontinuum generation

② High rep-rate ultrashort light pulse generation.





More coherent

2. Optical Solitons

$\beta_2 < 0$, MI can occur.

$\Rightarrow$ pulse-like solution, do not change during propagation!

Start w/ NLS:

$$\frac{\partial A}{\partial z} + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} = i\gamma |A|^2 A.$$

Dimensionless quantities:

$$U = \frac{A}{\sqrt{P_0}}, \quad \tau = \frac{T}{T_0}, \quad \xi = \frac{z}{L_D} \qquad (T_0 \equiv \text{pulse width})$$

$$\Rightarrow \quad i\frac{\partial u}{\partial \xi} = \overset{= -1}{\operatorname{sgn}(\beta_2)}\,\frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - N^2 |u|^2 u.$$

where $N^2 = \dfrac{L_D}{L_{NL}} = \dfrac{\gamma P_0 T_0^2}{|\beta_2|}$ $\quad \left(L_D \to \dfrac{T_0^2}{|\beta_2|},\ L_{NL} \sim \dfrac{1}{\gamma P_0}\right)$

let $u = N \cdot U = \sqrt{\gamma L_D}\cdot A$, NLS becomes

$$\boxed{i\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0} \qquad (\beta_2 < 0)$$

How to solve? Inverse scattering transform. (Chapter 5. Agrawal)

① When $N \ll 1$, i.e. $L_D \ll L_{NL}$

Dispersion is dominant. pulse gets broadened!

② When $N = 1$:

Solution: "Fundamental soliton."

$$u(\xi, \tau) = \eta\, \mathrm{sech}(\eta\tau)\, \exp(i\eta^2 \xi/2) \qquad \mathrm{Sech}(x) = \frac{2}{e^{x} + e^{-x}}$$

η = arbitrary parameter. determines both amplitude and width

time width of soliton $= \dfrac{T_0}{\eta}$.

const.

Dimensionless unit: $\mathrm{amp}^2 \cdot \times \mathrm{width}^2 = \eta^2 \cdot \dfrac{1}{\eta^2} = 1$

Dimension full unit: $\mathrm{amp}^2 \times \mathrm{width}^2 = P_0 \cdot \eta^2 \times \dfrac{T_0^2}{\eta^2} = P_0 T_0 = \boxed{\dfrac{|\beta_2|}{\gamma}}$

Conclusion: For solitons, width is inversely proportional to amplitude!

"solitary wave": pulse propagates w/ fixed shape

"soliton": solitary wave that retains its shape after "collisions".

What if we don't start with the perfect shape, or N exactly equals to 1 ?

$\Rightarrow$ get fundamental solitons ($N=1$) plus some dispersive loss! See slides.

③ When $N \geq 2$

$\Rightarrow$ higher-order solitons.

$$u(\xi,\tau) = \frac{4\left[\cosh(3\tau) + 3\exp(4i\xi)\cosh(\tau)\right]\exp\left(\frac{i\xi}{2}\right)}{\left[\cosh(4\tau) + 4\cosh(2\tau) + 3\cosh(4\xi)\right]}$$

$|u(\xi,\tau)|^2$ has a peridicity of $\xi_0 = \left(\frac{z}{L_D}\right) = \frac{\pi}{2}$.

$$\Rightarrow z = \frac{\pi}{2} L_D = \frac{\pi}{2}\frac{T_0^2}{|\beta_2|} \simeq \frac{T_{FWHM}^2}{2|\beta_2|} \quad (\text{"soliton peridicity"})$$

Evolution $\Rightarrow$ see slides!

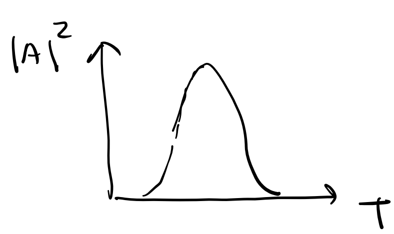
Application $\Rightarrow$ Supercontinuum generation (soliton fission)

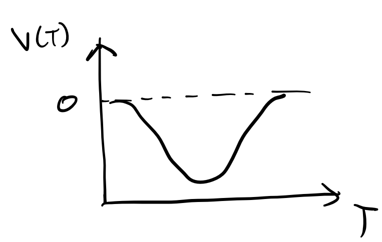
3. Physical interpretation of solitons.

Schrodinger eq: $$i\hbar \frac{\partial \psi}{\partial T} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

NLS eq: $$i\frac{\partial A}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \boxed{\gamma |A|^2} A$$

nonlinear potential $V(T)$ created by the pulse itself





Soliton : pulse trapped by the nonlinear potential created by itself!