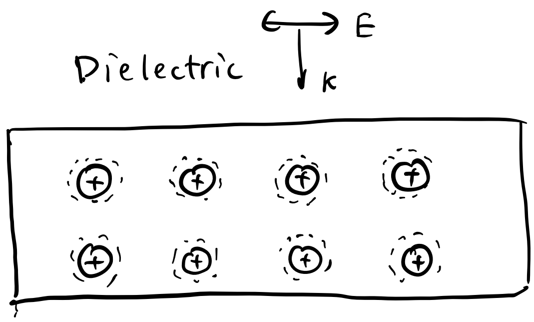
Lecture 10 Metal optics

Learning objectives:

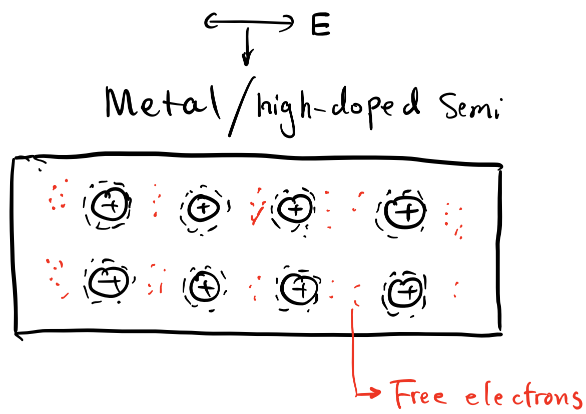
1. Free electrons
2. The Drude model.
3. Dispersion of the free electron gas and bulk plasmons.

1. Free electrons

Dielectric



Metal / high-doped Semi



Restoring force

⇒ Natural resonant freq.

No restoring force

⇒ No resonant freq.

Oscillations of free electrons induced by AC electric field.

$$\underbrace{m_0 \frac{d^2x}{dt^2}}_{\text{acceleration}} + \underbrace{m_0 \gamma \frac{dx}{dt}}_{\text{damping}} = -eE(t) = \underbrace{-eE_0 e^{-i\omega t}}_{\text{drive}} \qquad (1)$$

Plug in $x = x_0 e^{-i\omega t}$,

$$\boxed{x = \frac{eE}{m_0(\omega^2 + i\gamma\omega)}}$$

Recall, for bound electrons. $\quad x = \dfrac{-eE}{m_0(\omega_0^2 - \omega^2 - i\gamma\omega)}$

Polarizability $P = -Nex = \dfrac{-Ne^2 E}{m_0(\omega^2 + i\gamma\omega)}$

N: # of electrons per unit volume

$$D = \epsilon_r \epsilon_0 E = \epsilon_0 E + P = \epsilon_0 E - \frac{Ne^2 E}{m_0(\omega^2 + i\gamma\omega)}$$

$$\Rightarrow \epsilon_r(\omega) = 1 - \frac{Ne^2}{\epsilon_0 m_0} \frac{1}{(\omega^2 + i\gamma\omega)}$$

$$\Rightarrow \epsilon_r(\omega) = 1 - \frac{\omega_p^2}{(\omega^2 + i\gamma\omega)}$$

where $\omega_p = \sqrt{\dfrac{Ne^2}{\epsilon_0 m_0}}$ is known as plasma freq. of the free electron gas.

Comments:

1. In metals, $N \sim 10^{28} \sim 10^{29}\ m^{-3}$, ω_p is in UV region
2. $\gamma = \frac{1}{\tau} = 10^{13} \sim 10^{14}$.
3. When $\omega > \omega_p$, $\omega\ (>10^{15}\,Hz) \gg \gamma$,

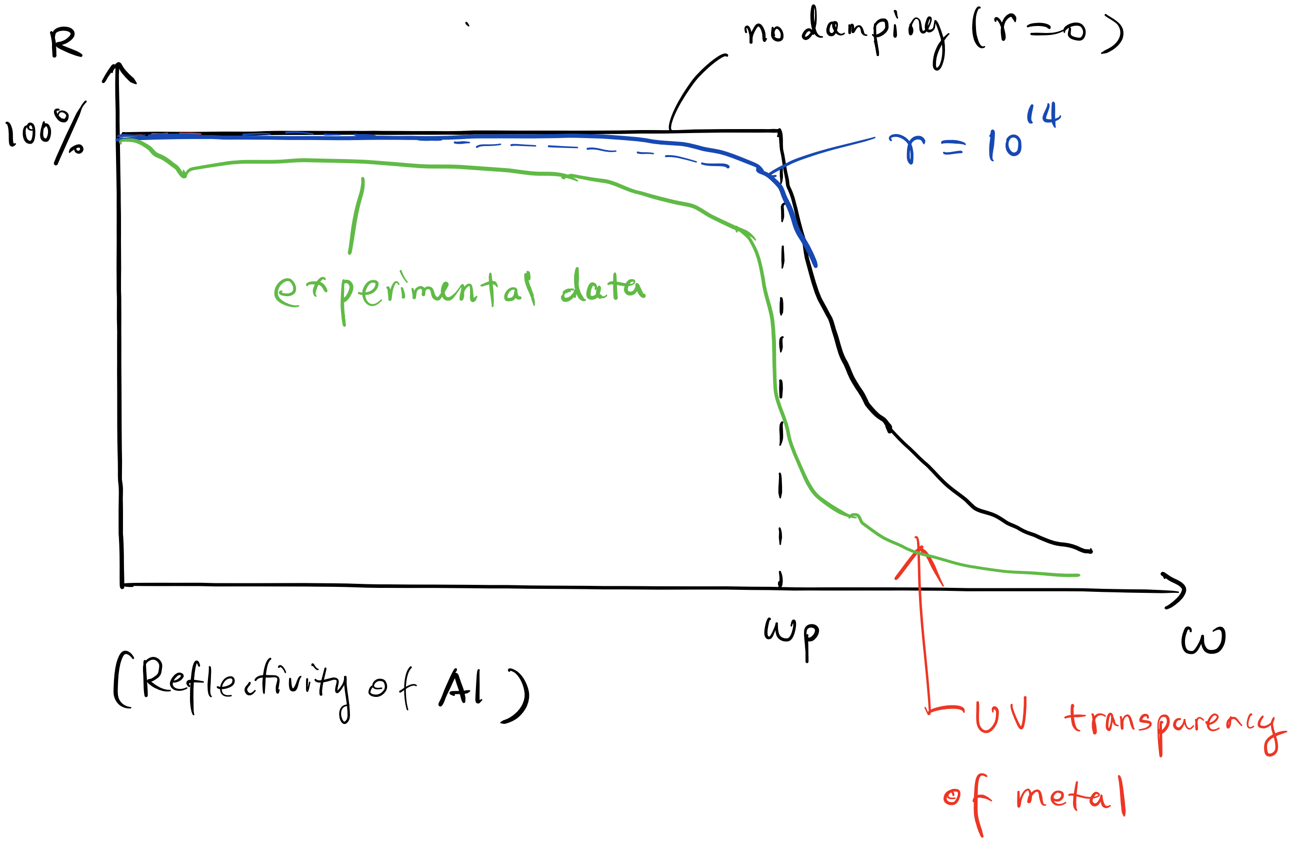
$$\tilde{n} \simeq \sqrt{\epsilon_r(\omega)} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad \text{(real number)}$$

when $\omega = \omega_p$. $\omega \gg \gamma$, $\tilde{n} = 0$

when $\omega < \omega_p$. $\tilde{n}$ is complex.

For normal incidence,

$$R = \left| \frac{\hat{n} - 1}{\hat{n} + 1} \right|^2$$

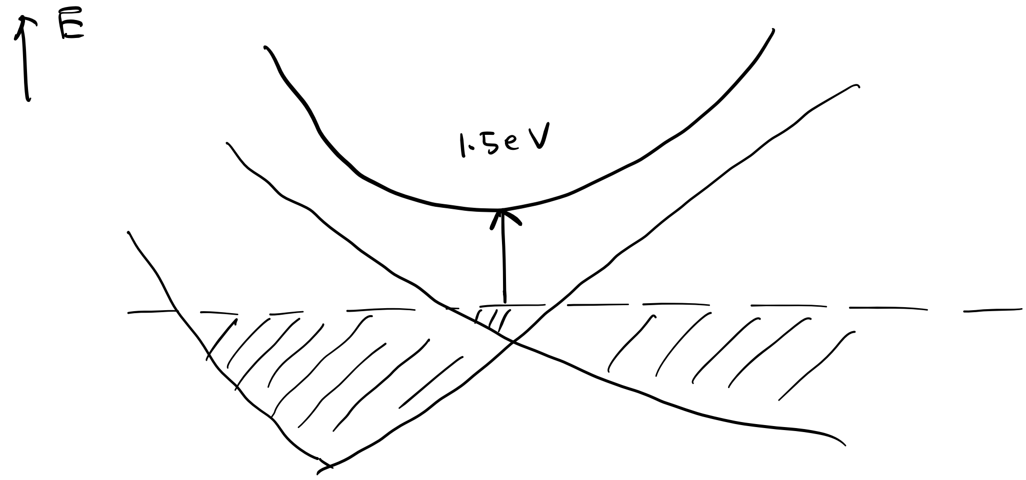


(Reflectivity of Al)

Reasons for non-unity reflection in metal:

Interband transition

Al: parallel band



For Al. $E_{interband} \simeq 1.5$ eV

Ag	..	(UV) 4 eV	(good mirror)
Cu	..	2 eV.	(reddish color)
Au	...	.. > 2 eV	(brighter than copper. good mirror for vis)

2. Free carrier conductivity (Drude model)

Eq. ① can be rewritten as

$$m_0 \frac{d\vec{v}}{dt} + m_0 \gamma \vec{v} = -e\vec{E}$$

use $p = m_0 v$. (momentum) and $\gamma = \frac{1}{\tau}$, where τ is the damping time, we have

$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e\vec{E}$$

With AC $\vec{E}$-field drive, $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$, $\vec{v} = \vec{V}_0 e^{-i\omega t}$

$$\Rightarrow \quad \vec{v}(t) = \frac{-e\tau}{m_0} \frac{1}{1 - i\omega\tau} \vec{E}(t)$$

The current density:

$$j = -Nev = \overset{\text{conductivity}}{\sigma} \vec{E}$$

$\Rightarrow$ AC conductivity:

Drude model.

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}} \qquad (2)$$

were $\sigma_0 = \dfrac{Ne^2\tau}{m_0}$ is the DC conductivity.

Recall that:

$$\epsilon_r = 1 - \frac{Ne^2}{\epsilon_0 m_0} \frac{1}{(\omega^2 + i\gamma\omega)} \qquad (3)$$

Compary (2) and (3),

$$\boxed{\epsilon_r(\omega) = 1 + \frac{i\sigma(\omega)}{\epsilon_0 \omega}}$$

Comments:

τ can be determined by measuring DC conductivity (σ_0)

For metals, $\tau \sim 10^{-14} \sim 10^{-13}$

For metals, $\epsilon_r(\omega)$ and $\sigma(\omega)$ are correlated.

$$\epsilon_r(\omega) = \epsilon_1(\omega) + i\,\epsilon_2(\omega).$$

$$\epsilon_1 = 1 - \frac{\omega_p^2 \tau^2}{1+\omega^2\tau^2}$$

$$\epsilon_2 = \frac{\omega_p^2 \tau}{\omega(1+\omega^2\tau^2)}$$

① At very low frequency, $\omega\tau \ll 1$ (i.e. $\omega \ll \frac{1}{\tau}$)

$\epsilon_2 \gg \epsilon_1$.

$$n = \frac{1}{\sqrt{2}}\sqrt{\epsilon_1 + \sqrt{\epsilon_1^2+\epsilon_2^2}}$$

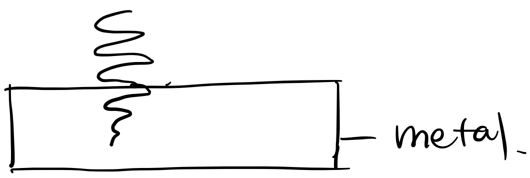
$$k = \frac{1}{\sqrt{2}}\sqrt{-\epsilon_1 + \sqrt{\epsilon_1^2+\epsilon_2^2}}$$

$$\Rightarrow n \simeq k = \sqrt{\frac{\epsilon_2}{2}}$$

$$\boxed{\alpha = \frac{4\pi k}{\lambda} = \sqrt{\frac{2\omega_p^2\tau\omega}{c^2}} \overset{\omega_p^2\tau = \frac{\sigma_0}{\epsilon_0}\quad c^2=\frac{1}{\epsilon_0\mu_0}}{=\!=} \sqrt{2\sigma_0\omega\mu_0}}$$

Physical meaning:

1) AC electric field can only penetrate a short distance in to a metal. (a.k.a Skin effect)



2) Define δ as the skin depth.

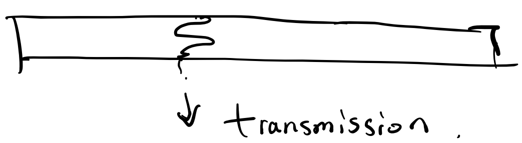
E-field decays as $\exp(-\frac{z}{\delta})$, power decays as $\exp(-\frac{2z}{\delta})$. Recall the definition of α: $I(z) = I_0 e^{-\alpha z}$.

$$\boxed{\delta = \frac{2}{\alpha} = \sqrt{\frac{2}{\sigma_0 \omega \mu_0}}} \qquad (4)$$

3) The decaying field in metal is evanescent waves.

4) When thickness of metal is comparable or smaller than δ, evanescent wave will not decay fully.

R is not 100%.



5) eq. ④ only valid when $\omega\tau \ll 1$. !

(low frequency!)

② <u>At higher frequency ($1 \leq \omega\tau \leq \omega_p \tau$)</u>

$\tilde{n} = n + ik$ is predominantly imaginary.

$$R = \left(\frac{\tilde{n}-1}{\tilde{n}+1}\right)^2 \simeq 1$$

③ At very high frequency. ($\omega \gtrsim \omega_p$, $\omega\tau \gg 1$)

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{(damping is negligible)} \quad ①$$

3. Dispersion of free electron gas and bulk plasmons

When $\omega = \omega_p$, eq. ① becomes zero. ⇒ something interesting!

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

take the time derivative :

$$\frac{\partial \vec{J}}{\partial t} + \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0} \nabla \times \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{1}{\mu_0} \nabla \times (\nabla \times \vec{E}) \quad ①$$

the equation of motion of electrons :

$$m \frac{d\vec{v}}{dt} = -e\vec{E}$$

also, $J = -Nev.$

$$\Rightarrow \quad \frac{\partial \vec{J}}{\partial t} = \frac{Ne^2}{m} \vec{E} \qquad (2)$$

plug in (2) into (1), use $\omega_p = \sqrt{\frac{Ne^2}{m\epsilon_0}}$, $c^2 = \frac{1}{\mu_0 \epsilon_0}$.

$$\frac{\partial^2 \vec{E}}{\partial t^2} + \omega_p^2 \vec{E} = -c^2 \nabla \times (\nabla \times \vec{E}) \qquad (3)$$

Split $\vec{E}$ into transverse and longitudinal components. i.e.

i.e. $\vec{E} = \vec{E}_t + \vec{E}_l$, $\nabla \cdot \vec{E}_t = 0$; $\nabla \times \vec{E}_l = 0$

$$\frac{\partial^2 \vec{E}_t}{\partial t^2} + \omega_p^2 \vec{E}_t - c^2 \nabla^2 \vec{E}_t = -\left(\frac{\partial^2 E_l}{\partial t^2} + \omega_p^2 E_l \right)$$

$$\Rightarrow \begin{cases} \frac{\partial^2 \vec{E}_t}{\partial t^2} + \omega_p^2 E_t - c^2 \nabla^2 \vec{E}_t = 0 & \text{(transverse)} \\ \frac{\partial^2 E_l}{\partial t^2} + \omega_p^2 E_l = 0 & \text{(longitudinal)} \end{cases}$$

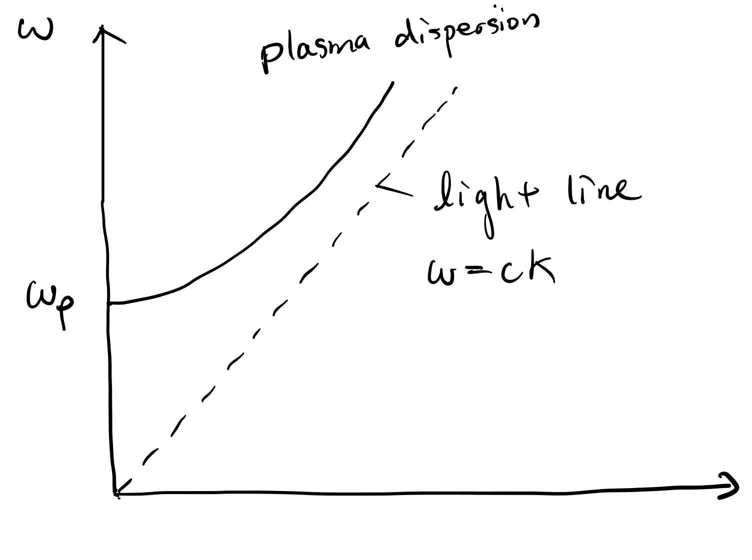
For the transverse wave:

$$c^2 k^2 = \omega^2 - \omega_p^2$$

For the longitudinal mode

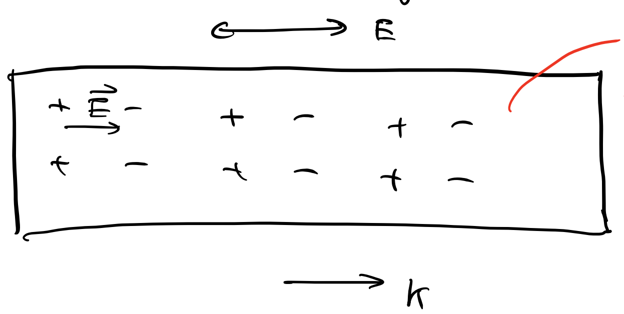
$$\omega = \omega_p.$$

Dispersion relation of the free electron gas



① . when $\omega > \omega_p$. transverse wave ($\vec{k} \cdot \vec{E} = 0$) can propagate in metals.

② when $\omega = \omega_p$, dispersion less (ω is independent of $\vec{k}$) longitudinal mode, corresponding to plasma oscillations



quanta of these charge oscillation are called plasmons.

③ when $\omega < \omega_p$, no mode. transverse wave cannot propagate in metal. they are reflected by the plasma in metal.