Lecture 1. Review of Maxwell equations

<u>Learning Objectives:</u>

1. Review of maxwell equations.
2. Maxwell equations in vaccum
3. Maxwell equations in materials

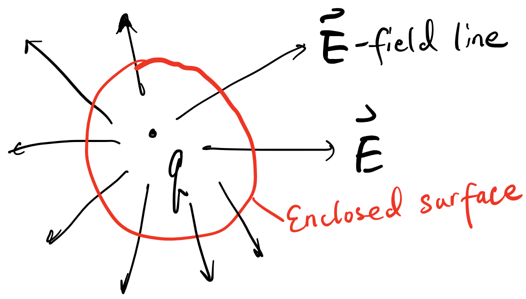
1. Review of Maxwell equations.

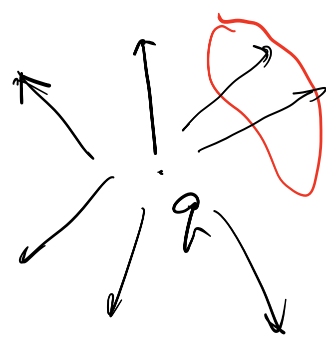
Gauss's Law
Faraday's Law
Ampere's Law
No magnetic charge

$\Rightarrow$ Maxwell eqs. $\Rightarrow$

① E&M wave equations
② light is E&M waves!

① Gauss's Law (P66. Griffith)





Flux of $\vec{E}$ through a surface:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

— total charge enclosed

Physical meaning:

① The flux through any surface enclosing the charge is $\frac{q}{\epsilon_0}$

② Here, we are discussing free charge. In materials, we have bound charge. (Will be discussed later)

Applying the divergence theorem.

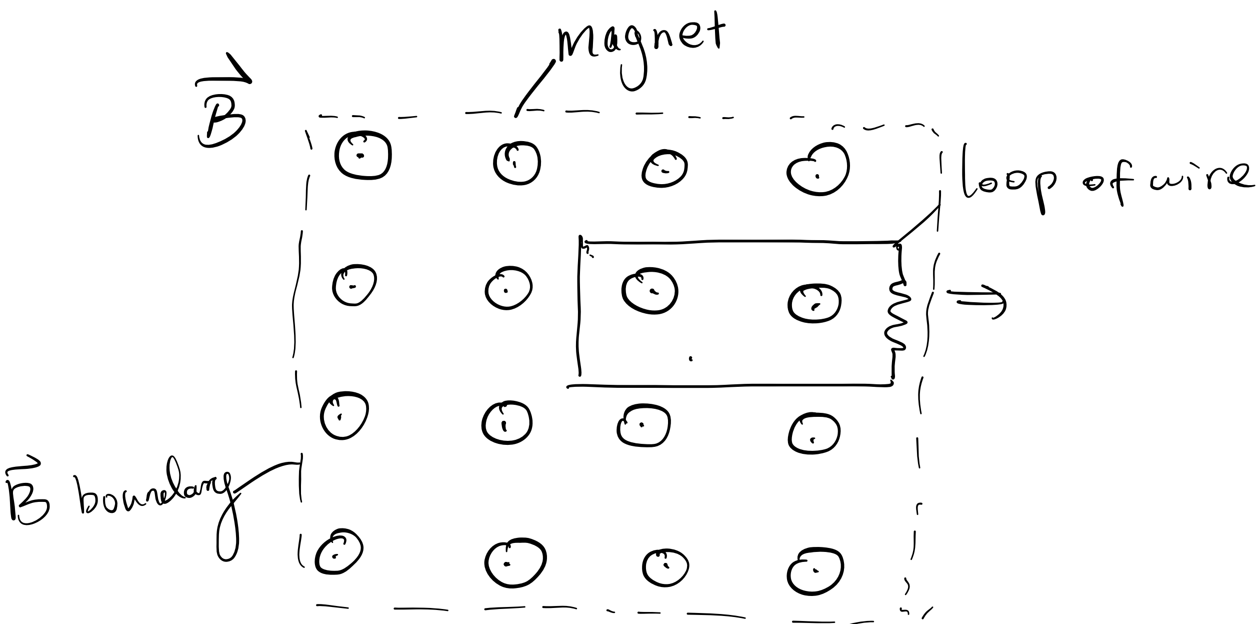
$$\oint_S \vec{E} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{E}) \cdot dV$$

then $\int_V (\nabla \cdot \vec{E}) \cdot d = \frac{Q_{enc}}{\epsilon_0} = \frac{\int_V \rho\, dV}{\epsilon_0}$

$$\Rightarrow \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$ where ρ is the charge density.

② Faraday's Law (P312, Griffith)

Derived from experiments.



experient 1:

move the loop of wire to the right.

$\Rightarrow$ a current flow in the loop.

experiment 2:

move the magnet to the left while holding the loop still.

$\Rightarrow$ a current flow in the loop

experiment 3:

change the strength of the $\vec{B}$,

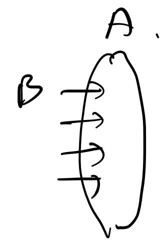
$\Rightarrow$ a current flow.

Faraday has an ingeneous inspiration:

$$\boxed{\text{A changing } \Phi \text{ will induce } \vec{E} .}$$

voltag (potential) $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\dfrac{d\Phi}{dt}$ → magnetic flux

Flux $\Phi = \int_A \vec{B} \cdot d\vec{A}$



Faraday's Law becomes

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial B}{\partial t} \cdot d\vec{A}$$

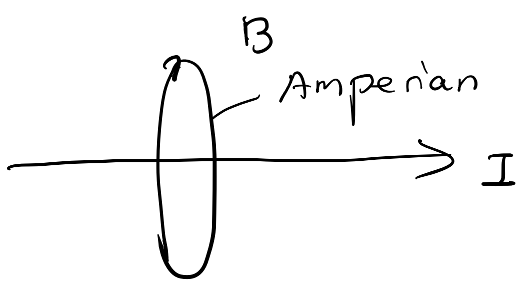
using stoke's theorem,

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$\vec{B}$: magnetic flux density vector (T)

③ Ampere's Law (P233 Griffith)

(Derived from experiment)



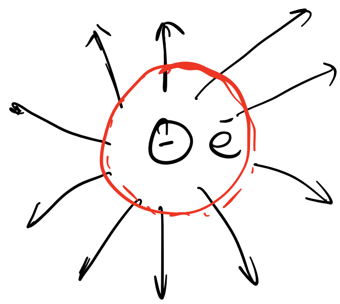
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

the current enclosed by the Amperian loop

use Stoke's theorem.

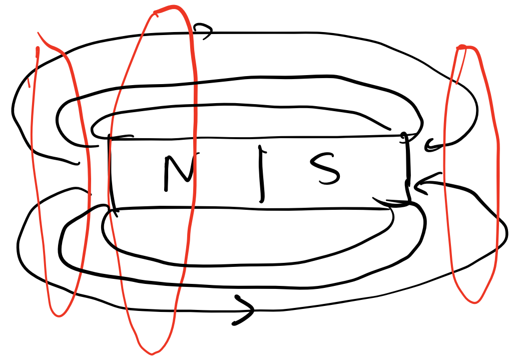
$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$ where $J = \frac{I_{enc}}{A}$ is the current density.

④ no magnetic charge ?



Electric field flux can be nonzero

$$\nabla \cdot \vec{E} = \frac{q}{\epsilon_0}$$



Magnetic flux is alway zero

$$\nabla \cdot \vec{B} = 0$$

(No magnetic monopole)

Now, we have four equations:

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{q}{\epsilon_0} & ① \text{ (Gauss's Law)} \\ \nabla \cdot \vec{B} = 0 & ② \text{ (No name)} \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & ③ \text{ (Faraday's Law)} \\ \nabla \times \vec{B} = \mu_0 \vec{J} & ④ \text{ (Ampere's Law)} \end{cases}$$

So, what is Maxwell's contribution ?

Mathematically,

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\nabla \cdot (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \cdot B) = 0 \quad \checkmark$$

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 (\nabla \cdot \vec{J}) \quad ?$$

if current is spatially uniform, fine

if current is spatially non-uniform, it is wrong.

Recall the continuity eq. (Derivation is in page 222 of Griffith)

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \vec{E}) = -\nabla \cdot \left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

So, to make ④ correct, we need to have

$$\nabla \times \vec{B} = \mu_0 J + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

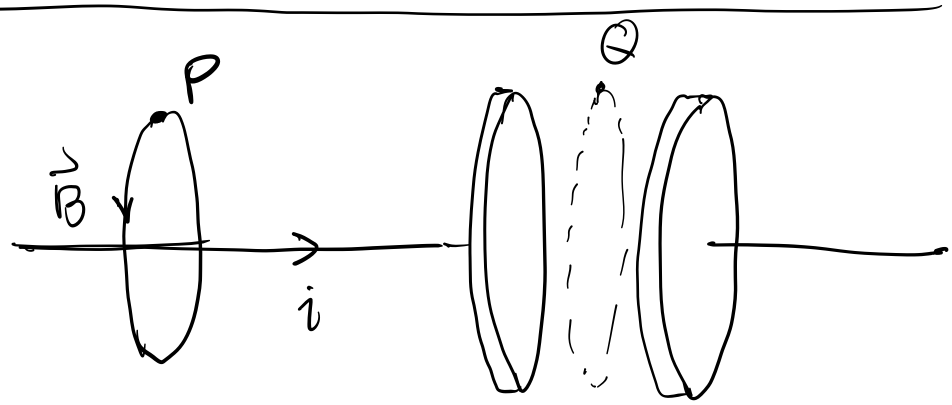
add this quantity to Ampére's Law.

this quantity is called displacement current.

Physical meaning:

A varying $\vec{E}$ can also induce $\vec{B}$.

Why Ampere's Law is problemetic?



At point P, $\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$

At ponit P, $\oint \vec{B} \cdot d\vec{l} = 0$, no current flow.

But according to experiments: $\vec{B}_P = \vec{B}_Q$

How to reconcile this paradox?

introduce displacement current.

$$J_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ (virtual Current)}$$

2. Maxwell equations in vacuum

$$\begin{cases} \nabla\cdot\vec{E} = \dfrac{q}{\epsilon_0} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{B} = \mu_0(J+J_d) = \mu_0\left(J+\epsilon_0\dfrac{\partial E}{\partial t}\right) \end{cases}$$

($\epsilon_0 = 8.85\times10^{-12}$ F/m, permittivity of vacuum)

($\mu_0 = 4\pi\times10^{-7}$ H/m, permeability of vacuum)

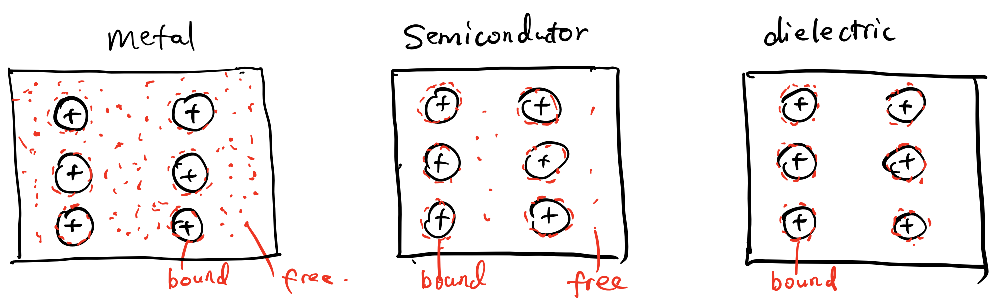
Comments:

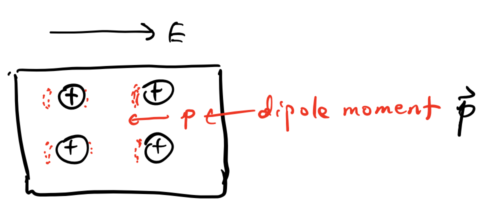
if we set $q=0$, $J=0$, (no source), non-zero solution of $\vec{E}$ and $\vec{B}$ exist, meaning EM field can exist even in the absence of any charge or current. This is something called EM waves.

3. Maxwell eq. in matters (Griffith. P167)

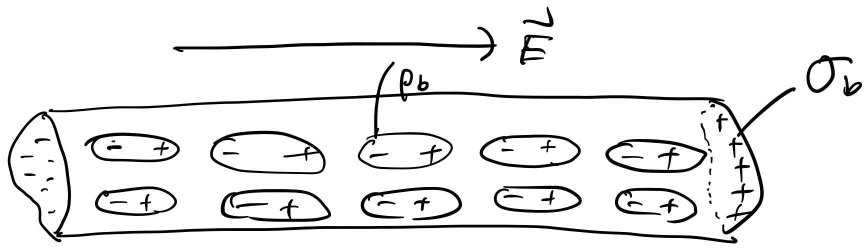
So far, we only discussed E&M in vacuum, what about in materials?

Microscopic picture of materials:





Imagine we have a dielectric chunck



resuts:

1) We have a lot of dipole moments within the chunck
2) We have a lot of surface charges accumulated

Define $\vec{P} \equiv$ dipole moment per unit volume

$\sigma_b \equiv$ area density of surface bound charges

$\rho_b \equiv$ volume density of bound charges inside matter.

Assume total charge q at surfaces.

$$\sigma_b = \frac{q}{A} \qquad (1)$$

total dipole moment of the structure:

$$qd = P \cdot Ad \qquad (2)$$

$$\Rightarrow \quad q = PA$$

Plug (2) in to (1), and in general

$$\sigma_b = \vec{P} \cdot \vec{n} \quad , \text{ where } \vec{n} \text{ is perpendicular to } A$$

Charge neutrality of the material requires:

$$\int \rho_b \, dV + \int \sigma_b \, dA = 0$$

$$\Rightarrow \int \rho_b \, dV = - \int \sigma_b \, dA = - \int \vec{P} \cdot d\vec{A} = - \int \vec{\nabla} \cdot \vec{P} \, dV$$

$$\Rightarrow \quad \rho_b = - \vec{\nabla} \cdot \vec{P}$$

Then Gauss's Law in the presence of materials is

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_b + \rho_f}{\epsilon_0} = \frac{- \nabla \cdot \vec{P} + \rho_f}{\epsilon_0}$$

$$\Rightarrow \quad \nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

Define $\vec{D}$ as the electric displacement vector

$$\nabla \cdot \vec{D} = \rho_f \quad , \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Importance: it makes references only to free charges!

Bound charge is not involved!

In linear dielectrics:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

χ_e: electric susceptibility or polarizability.

which measn that the induced $\vec{P} \sim \vec{E}$.

when the $\vec{E}$ field is very large,

$$\vec{P} = \epsilon_0 \left(\chi^{(1)} \vec{E} + \chi^{(2)} E^2 + \chi^{(3)} E^3 \cdots\cdots \right)$$

the $\vec{P}$ is no longer linearly depends on $\vec{E}$.

(study of nonlinear optics...)

In the linear case:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon_0 \epsilon_r \vec{E}$$

Define $\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$ as relative permittivity.

Similarly, in magnetic materials.

Define $\vec{H} = \frac{1}{\mu_0}\vec{B} - M$

Magnetization vector

$$M = \chi_m \frac{1}{\mu_0} B$$

magnetic susceptibility.

$$\Rightarrow \vec{H} = \frac{\vec{B}}{\mu \mu_0}$$

In this coure, we won't discuss magnetic materials

$(M=0,\ \chi_m = 0,\ \mu = 1)$

⑥ General Maxwell equations

$$\begin{cases} \nabla \cdot \vec{D} = \rho_{free} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{cases}$$

$\vec{D}$: electric displacement vector $\left(\frac{C}{m^2}\right)$

$\vec{B}$: magnetic flux density vector (W/m^2)

$\vec{E}$: electric field vector (V/m)

$\vec{H}$: magnetic field vector (A/m)

Constitutive equations (relates $\vec{D}$, $\vec{E}$ and $\vec{B}$, $\vec{H}$)

$$\begin{cases} D \equiv \epsilon_0 \epsilon_r E = \epsilon E \\ B \equiv \mu_0 \mu_r H = \mu H \end{cases}$$

Boundary Conditions (B.C.)

$$\begin{cases} \vec{E}^{\parallel}_{\text{above}} = \vec{E}^{\parallel}_{\text{below}} \\ \vec{D}^{\perp}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \sigma_s \\ \vec{B}^{\perp}_{\text{above}} = \vec{B}^{\perp}_{\text{below}} \\ \vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = K \end{cases}$$

← surface current density

above

below

Comment:

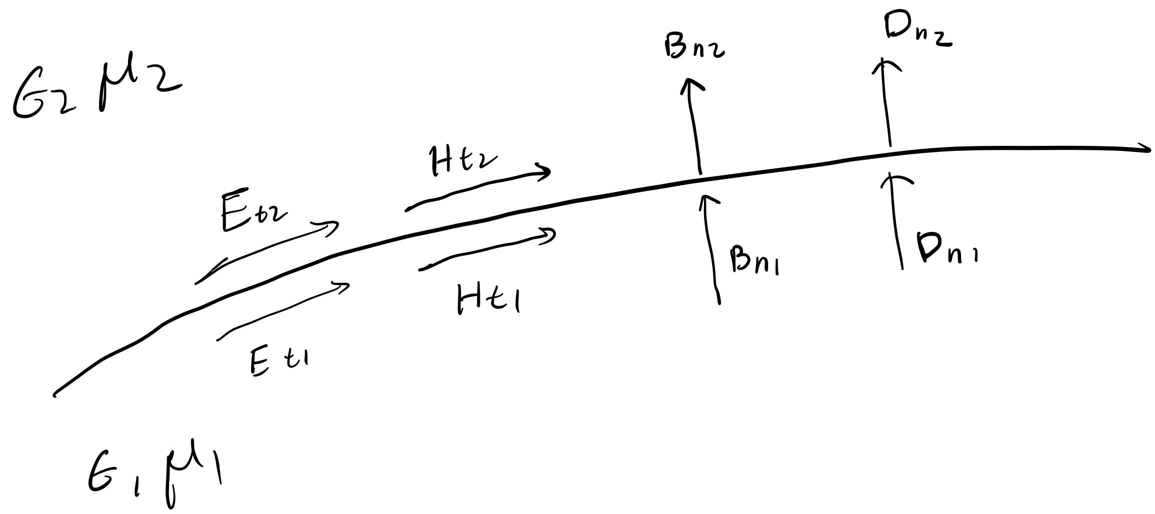
In most photonic materials:

$$\sigma_s = 0, \quad K = 0.$$

⇒ B.C.

① Tangential Component of $\vec{E}$, $\vec{H}$ are continous

② Normal component of $\vec{D}$, $\vec{B}$ are continous.

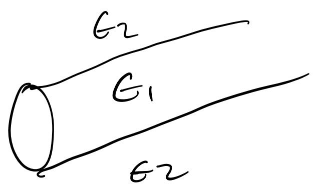


Comments:

Almost all photonic problems are about solving Maxwell equations with different boundary conditions,

e.g. waveguide, fiber, plasmonics, topological photonics, polaritons metamaterials/metasurfaces, solitons...

waveguide



Plasmonics